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Chapter 1

Basic concepts and equations

1.1 Introduction

This chapter includes some standard definitions, concepts and derivation of some basic equations for the analysis of fluid motion.

1.2 Definitions

Rotational flow

A flow in which the fluid particles have some angular velocity about their own angular axes, while flowing is called rotational flow. For such flow

$$curl\mathbf{V} \neq 0.$$

The presence of viscous forces means that the flow is rotational.

Irrotational flow

A flow which is not rotational. For this flow

$$curl\mathbf{V} = 0.$$

Fluid mechanics

The branch of science which deals the behavior of fluids at rest and in motion .

Static equilibrium

A fluid body is said to be in static equilibrium if the sum of all forces acting on it is zero.

Fluid Statics

Fluid statics is that branch of fluid mechanics which deals the fluid at rest. In fluid statistics the forces acting on the fluid are due to pressure and gravity only.

Fluid Dynamics

The branch of fluid mechanics which deals with the fluid in motion. Fluid dynamics can be divided into two categories.

Hydrodynamics

In hydrodynamics there is virtually no density change, i.e. flow of liquid or gas at low speeds

Gas dynamics

In gas dynamics fluids undergo significant density changes.

Types of fluids

A fluid can be either a gas or a liquid. i.e. compressible or incompressible.

Compressible flow

Flow in which variation within a flow are not negligible, the flow is called compressible. The most common example is flow of gases. For such flow the density changes.

Incompressible flow

Flow in which variation in density are negligible are termed as incompressible i.e. flow of liquid may frequently be treated as incompressible.

Ideal fluid

Ideal fluid is defined as one in which viscosity is zero. Such fluid does not exist in reality.

Real fluid

The fluids for which viscosity is not zero.

1.3 Properties of fluids

Viscosity

It is the resistance to flow when it is subjected to a shear stress. It is the measure of the internal friction that arises when there are velocity gradient with in the flow. Mathematically

$$\mu = \frac{shear\ stress}{Velocity\ gradient}.$$

It is also called the coefficient of viscosity, the absolute viscosity, the dynamic viscosity or simply viscosity.

Kinematic viscosity

The ratio of absolute viscosity μ to density ρ , often arises .This ratio is called the kinematic viscosity and is given by

$$\nu = \frac{\mu}{\rho}.$$

Density

The density ρ is defined as the mass per unit volume. Mathematically

$$\rho = \frac{m}{V_1}.$$

in which m is the mass and V_1 is the volume.

1.3.1 Newtonian and Non-Newtonian fluids

Newtonian fluids

Fluids which obey the Newton's law of viscosity are called Newtonian fluids. Newton's law of viscosity is given by

$$\tau = \mu \frac{du}{dy},$$

where

$$\begin{aligned} \tau &= \text{shear stress,} \\ \mu &= \text{viscosity of fluid,} \\ \frac{du}{dy} &= \text{velocity gradient.} \end{aligned}$$

Gases and most common liquids tend to be Newtonian fluids. Most common examples are water, and mercury.

Non-Newtonian fluids

Fluids which do not obey the Newton's law of viscosity i.e. shear stress is not directly and linearly proportional to shear rate, are called non-Newtonian fluids. These fluids obey the power-law model. Power-law model is defined as

$$\tau = \eta \frac{du}{dy},$$

where the apparent viscosity is

$$\eta = k \left(\frac{du}{dy} \right)^{n-1}.$$

in which constant k is the consistency index and n is flow behavior index. When $n = 1$, the above equation reduces to Newton's law of viscosity with $k = \mu$. Examples include ketchup, chewing gum, shampoo, and mud.

1.3.2 A comparison of Newtonian and Non Newtonian fluids

The theory of non-Newtonain fluids based on the continuum hypothesis that a fluid particle may be regarded as continuous in structure. Non-Newtonian fluid are classified on the basis of their behavior in shear. Fluids for which shear stress is directly and linearly proportional to the deformation rate are Newtonian. In contrast to the definition of Newtonian fluid one can define non-Newtonian fluid as a fluid that exhibits non linear relationship between the shear stress and deformation rate. Non-Newtonian fluids are often divided into three broad groups, time dependent, time independent, and viscoelastic respectively. Exact analytic solutions for a Newtonain fluid are a few in number. These even become rare if constitutive equations for non-Newtonian fluids have been taken into account. One particular class of non-Newtonian fluids for which one can reasonably hope to obtain analytic solution is second grade fluid model. In this dissertation, we will consider this model.

Force

An agent which changes or tend to change the state of rest or motion of a body.

Body force

Force which is developed without physical contact and distributed over the volume of the

fluid are known as the body forces. The gravitational and electromagnetic forces are examples of body forces.

Surface force

These are the forces acting on the boundaries of a medium through direct contact. i.e. surface tension.

Inertial force

If several forces act on a particle of mass m and produce an acceleration a. Then by Newton's law

$$F - ma = 0,$$

in which $-ma$ is called the inertial force. Examples are the Coriolis force and the centrifugal force that appears in rotating coordinate system.

Coriolis force

The Coriolis force is the inertial force associated with a change of a tangential component of a particles, velocity. It is also define as a fictitious force exerted on a body when it moves in a rotating frame of reference.

Centrifugal force

The centrifugal force is an outward-directed fictitious force exerted on a body when it moves azimuthally in a non-inertial rotating frame of reference.

Magnetohydrodynamics

It is the subject that deals with the study of motion of fluid in the presence of magnetic field. The fluid under consideration is known as magnetohydrodynamic fluid or simply MHD fluid.

Electric field

Force acting on a unit charge is known as electric field i.e.

$$\mathbf{E} = \frac{F}{q_0}.$$

The direction of E is the same as that of F.

Magnetic field

Magnetic field commonly denoted by $\mathbf{B}$, are caused by currents or equivalently by the movement of charges. Usually the space around a permanent magnet or current carrying conductor is described as the location of the magnetic field. In particular the Maxwell equation in the absence of electric field gives

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_e \mathbf{J},$$

where u_e is the permeability of free space and $\mathbf{J}$ is the current density.

Current density

The quantity of charge passing through unit area of cross section of a conductor per second is called current density. It is a vector quantity.

Permeability

The ratio of the magnetic induction in the substance to the magnetizing field is called permeability.

Maxwell's equations

The Maxwell's equation are the set of four fundamental equations governing electromagnetism, i.e. the behavior of electric and magnetic fields. They were first given by James Clerk Maxwell. For the varying fields the differential form of these equation is given by

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_\circ}, \\
\boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\
\boldsymbol{\nabla} \cdot \mathbf{B} &= 0, \\
\boldsymbol{\nabla} \times \mathbf{B} &= \mu_e \mathbf{J} + \varepsilon_\circ \mu_e \frac{\partial \mathbf{E}}{\partial t},
\end{aligned}$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, $\mathbf{J}$ is the current density, $\varepsilon_\circ$ is the permittivity of free space and μ_e is the permeability of free space.

1.4 Basic Equations

The basic equations that enable us to predict fluid behavior are

1. Equation of continuity

2. Equation of motion

Equation of continuity

For a fluid region having no sources or sinks, the amount of fluid within the region is conserved. Mathematically, we can write

$$div(\rho \mathbf{V}) + \frac{\partial \rho}{\partial t} = 0, \tag{1.1}$$

in which ρ is the density. For incompressible fluid, the density is invariable and hence one obtains the following equation.

$$\nabla \cdot \mathbf{V} = 0 \tag{1.2}$$

as continuity equation.

Equation of motion

In rotating frame the equation of motion is

$$\rho \left[\frac{d\mathbf{V}}{dt} + 2\Omega \widehat{k} \times \mathbf{V} + \Omega \widehat{k} \times (\Omega \widehat{k} \times \mathbf{r}) \right] = \rho \mathbf{b}_1 + div \mathbf{T}. \tag{1.3}$$

In above equation V is the velocity vector, ρ is the density , Ω is the angular velocity, $\widehat{k}$ is a unit vector parallel to z-axis, $\mathbf{b}_1$ is the body force and $\mathbf{r}$ the radial coordinate given by

$$r^2 = x^2 + y^2.$$

The constitute equation for a second grade fluid is

$$\mathbf{T} = -p\mathbf{I} + \mu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2, \tag{1.4}$$

in which μ is the coefficient of viscosity, $-pI$ is the indeterminate spherical stress,

α_1 and α_2 are normal stress moduli satisfying $\mu \geq 0$, $\alpha_1 \geq 0$, $\alpha_1 + \alpha_2 = 0$ [13]. The kinematical tensors $\mathbf{A}_1$ and $\mathbf{A}_2$ are defined through

$$\mathbf{A}_1 = (grad \mathbf{V}) + (grad \mathbf{V})^T,$$

$$\mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1 (grad \mathbf{V}) + (grad \mathbf{V})^T \mathbf{A}_1,$$

where **grad** denotes the gradient operator and $\frac{d}{dt}$ indicate the material derivative. Inserting Eq. (1.4) into Eq. (1.3) we have

$$\rho\left[\frac{d\mathbf{V}}{dt}+2\Omega\widehat{k}\times\mathbf{V}+\Omega\widehat{k}\times(\Omega\widehat{k}\times\mathbf{r})\right]=div\left[-p\mathbf{I}+\mu\mathbf{A}_1+\alpha_1\mathbf{A}_2+\alpha_2\mathbf{A}_1^2\right]+\rho b_1.$$

We seek the velocity field of the form

$$\mathbf{V}=\left[u(z,t),\ v(z,t),\ w(z,t)\right],$$

and thus

$$\Omega\widehat{k}\times\mathbf{V}=\begin{bmatrix}-\Omega v\\ \Omega u\\ 0\end{bmatrix}, \tag{1.5}$$

$$\Omega\widehat{k}\times(\Omega\widehat{k}\times\mathbf{r})=\begin{bmatrix}-\Omega x^2\\ -\Omega y^2\\ 0\end{bmatrix}, \tag{1.6}$$

$$\mathbf{L}=grad\mathbf{V}=\begin{bmatrix}0 & 0 & \frac{\partial u}{\partial z}\\ 0 & 0 & \frac{\partial v}{\partial z}\\ 0 & 0 & \frac{\partial w}{\partial z}\end{bmatrix},$$

$$\mathbf{L}^T=(grad\mathbf{V})^T=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z}\end{bmatrix},$$

$$\mathbf{A}_1=\mathbf{L}+\mathbf{L}^T=\begin{bmatrix}0 & 0 & \frac{\partial u}{\partial z}\\ 0 & 0 & \frac{\partial v}{\partial z}\\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & 2\frac{\partial w}{\partial z}\end{bmatrix}, \tag{1.7}$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 0 & \frac{\partial^2 u}{\partial z \partial t} + w\frac{\partial^2 u}{\partial z^2} \\ & & +\frac{\partial u}{\partial z}\frac{\partial w}{\partial z} \\ 0 & 0 & \frac{\partial^2 v}{\partial z \partial t} + w\frac{\partial^2 v}{\partial z^2} \\ & & +\frac{\partial v}{\partial z}\frac{\partial w}{\partial z} \\ \frac{\partial^2 u}{\partial z \partial t} + w\frac{\partial^2 u}{\partial z^2} & \frac{\partial^2 v}{\partial z \partial t} + w\frac{\partial^2 v}{\partial z^2} & 2\frac{\partial^2 w}{\partial z \partial t} + 2w\frac{\partial^2 w}{\partial z^2} + \\ +\frac{\partial u}{\partial z}\frac{\partial w}{\partial z} & +\frac{\partial v}{\partial z}\frac{\partial w}{\partial z} & 4(\frac{\partial w}{\partial z})^2 + 2\{(\frac{\partial u}{\partial z})^2 + (\frac{\partial v}{\partial z})^2\} \end{bmatrix},$$

$$\mathbf{A}_1^2 = \begin{bmatrix} (\frac{\partial u}{\partial z})^2 & \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} & 2\frac{\partial u}{\partial z}\frac{\partial w}{\partial z} \\ \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} & (\frac{\partial v}{\partial z})^2 & 2\frac{\partial v}{\partial z}\frac{\partial w}{\partial z} \\ 2\frac{\partial u}{\partial z}\frac{\partial w}{\partial z} & 2\frac{\partial v}{\partial z}\frac{\partial w}{\partial z} & 4\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \\ & & +\left(\frac{\partial v}{\partial z}\right)^2 \end{bmatrix},$$

$$pI = \begin{bmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{bmatrix}.$$

Using above equations, the Eq. (1.4) for Cauchy stress becomes

$$T = \begin{bmatrix} & & \mu\frac{\partial u}{\partial z} \\ -p + \alpha_2(\frac{\partial u}{\partial z})^2 & \alpha_2\frac{\partial u}{\partial z}\frac{\partial v}{\partial z} & +2\alpha_2\frac{\partial u}{\partial z}\frac{\partial w}{\partial z} + \alpha_1\frac{\partial^2 u}{\partial z \partial t} \\ & & +w\alpha_1\frac{\partial^2 u}{\partial z^2} + \alpha_1\frac{\partial u}{\partial z}\frac{\partial w}{\partial z} \\ & & \mu\frac{\partial v}{\partial z} \\ \alpha_2\frac{\partial u}{\partial z}\frac{\partial v}{\partial z} & -p + \alpha_2(\frac{\partial v}{\partial z})^2 & +2\alpha_2\frac{\partial v}{\partial z}\frac{\partial w}{\partial z} + \alpha_1\frac{\partial^2 v}{\partial z \partial t} \\ & & +w\alpha_1\frac{\partial^2 v}{\partial z^2} + \alpha_1\frac{\partial v}{\partial z}\frac{\partial w}{\partial z} \\ \mu\frac{\partial u}{\partial z} & \mu\frac{\partial v}{\partial z} & 2\mu\frac{\partial w}{\partial z} - p \\ +2\alpha_2\frac{\partial u}{\partial z}\frac{\partial w}{\partial z} & +2\alpha_2\frac{\partial v}{\partial z}\frac{\partial w}{\partial z} & +4\alpha_1(\frac{\partial w}{\partial z})^2 + 2\alpha_1\frac{\partial^2 w}{\partial z \partial t} \\ +\alpha_1\frac{\partial^2 u}{\partial z \partial t} & +\alpha_1\frac{\partial^2 v}{\partial z \partial t} & +2\alpha_1 w\frac{\partial^2 w}{\partial z^2} + 2\alpha_1\left(\frac{\partial u}{\partial z}\right)^2 \\ +w\alpha_1\frac{\partial^2 u}{\partial z^2} & +w\alpha_1\frac{\partial^2 v}{\partial z^2} & +2\alpha_1\left(\frac{\partial v}{\partial z}\right)^2 + 4\alpha_2\left(\frac{\partial w}{\partial z}\right)^2 \\ +\alpha_1\frac{\partial u}{\partial z}\frac{\partial w}{\partial z} & +\alpha_1\frac{\partial v}{\partial z}\frac{\partial w}{\partial z} & +\alpha_2\left(\frac{\partial u}{\partial z}\right)^2 + \alpha_2\left(\frac{\partial v}{\partial z}\right)^2 \end{bmatrix}.$$

From Eqs. (1.1) and (1.2)

$$\frac{\partial w}{\partial z} = 0.$$

The above equation implies that w is not a function of z. So either w is a function of t or constant. For uniform suction or blowing w is a constant i.e.

$$w = -W_0, \tag{1.7}$$

where $W_0 > 0$ corresponds to the suction velocity and $W_0 < 0$ corresponds to blowing. From Eq. (1.7) we have

$$T = \begin{bmatrix} -p + \alpha_2(\frac{\partial u}{\partial z})^2 & \alpha_2 \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} & \mu\frac{\partial u}{\partial z} + \alpha_1(\frac{\partial^2 u}{\partial z \partial t} - W_0\frac{\partial^2 u}{\partial z^2}) \\ \alpha_2 \frac{\partial u}{\partial z}\frac{\partial v}{\partial z} & -p + \alpha_2(\frac{\partial v}{\partial z})^2 & \mu\frac{\partial v}{\partial z} + \alpha_1(\frac{\partial^2 v}{\partial z \partial t} - W_0\frac{\partial^2 v}{\partial z^2}) \\ \mu\frac{\partial u}{\partial z} + & \mu\frac{\partial v}{\partial z} + & \\ \alpha_1(\frac{\partial^2 u}{\partial z \partial t} - W_0\frac{\partial^2 u}{\partial z^2}) & \alpha_1(\frac{\partial^2 v}{\partial z \partial t} - W_0\frac{\partial^2 v}{\partial z^2}) & -p + (2\alpha_1 + \alpha_2)\left(\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right) \end{bmatrix},$$

or

$$(div\mathbf{T})_x = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial z^2} + \alpha_1\frac{\partial}{\partial z}(\frac{\partial^2 u}{\partial z \partial t} - W_0\frac{\partial^2 u}{\partial z^2}),$$

$$(div\mathbf{T})_y = -\frac{\partial p}{\partial y} + \mu\frac{\partial^2 v}{\partial z^2} + \alpha_1\frac{\partial}{\partial z}(\frac{\partial^2 v}{\partial z \partial t} - W_0\frac{\partial^2 v}{\partial z^2}),$$

$$(div\mathbf{T})_z = -\frac{\partial p}{\partial z} + (2\alpha_1 + \alpha_2)\frac{\partial}{\partial z}\left(\left(\frac{\partial u}{\partial z}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2\right).$$

In scalar form Eq. (1.3) gives

$$\rho\left[\frac{d\mathbf{V}}{dt} + 2\boldsymbol{\Omega}\widehat{k} \times \mathbf{V} + \boldsymbol{\Omega}\widehat{k} \times (\boldsymbol{\Omega}\widehat{k} \times \mathbf{r})\right]_x = \rho b_{1x} + (div\mathbf{T})_x, \tag{1.8}$$

$$\rho\left[\frac{d\mathbf{V}}{dt} + 2\boldsymbol{\Omega}\widehat{k} \times \mathbf{V} + \boldsymbol{\Omega}\widehat{k} \times (\boldsymbol{\Omega}\widehat{k} \times \mathbf{r})\right]_y = \rho b_{1x} + (div\mathbf{T})_y, \tag{1.9}$$

$$\rho\left[\frac{d\mathbf{V}}{dt} + 2\boldsymbol{\Omega}\widehat{k} \times \mathbf{V} + \boldsymbol{\Omega}\widehat{k} \times (\boldsymbol{\Omega}\widehat{k} \times \mathbf{r})\right]_z = \rho b_{1x} + (div\mathbf{T})_z. \tag{1.10}$$

With the help of Eqs. (1.5), (1.6) and (1.7) we get

$$\rho\left[\frac{d\mathbf{V}}{dt} + 2\boldsymbol{\Omega}\widehat{k} \times \mathbf{V} + \boldsymbol{\Omega}\widehat{k} \times (\boldsymbol{\Omega}\widehat{k} \times \mathbf{r})\right]_x = \frac{\partial u}{\partial t} - W_0\frac{\partial u}{\partial z} - 2\Omega v,$$

$$\rho\left[\frac{d\mathbf{V}}{dt} + 2\boldsymbol{\Omega}\widehat{k} \times \mathbf{V} + \boldsymbol{\Omega}\widehat{k} \times (\boldsymbol{\Omega}\widehat{k} \times \mathbf{r})\right]_y = \frac{\partial v}{\partial t} - W_0\frac{\partial v}{\partial z} + 2\Omega u,$$

$$\rho\left[\frac{d\mathbf{V}}{dt}+2\mathbf{\Omega}\widehat{k}\times\mathbf{V}+\mathbf{\Omega}\widehat{k}\times(\mathbf{\Omega}\widehat{k}\times\mathbf{r})\right]_z=0.$$

So the Eqs. (1.8), (1.9) and (1.10) now takes the form

$$\frac{\partial u}{\partial t}-W_0\frac{\partial u}{\partial z}-2\Omega v=-\frac{1}{\rho}\frac{\partial\widehat{p}}{\partial x}+\frac{\mu}{\rho}\frac{\partial^2 u}{\partial z^2}+\frac{\alpha_1}{\rho}(\frac{\partial^3 u}{\partial z^2\partial t}-W_0\frac{\partial^3 u}{\partial z^3})+b_{1x}, \tag{1.11}$$

$$\frac{\partial v}{\partial t}-W_0\frac{\partial v}{\partial z}+2\Omega u=-\frac{1}{\rho}\frac{\partial\widehat{p}}{\partial y}+\frac{\mu}{\rho}\frac{\partial^2 v}{\partial z^2}+\frac{\alpha_1}{\rho}(\frac{\partial^3 v}{\partial z^2\partial t}-W_0\frac{\partial^3 v}{\partial z^3})+b_{1y}, \tag{1.12}$$

$$0=-\frac{1}{\rho}\frac{\partial\widehat{p}}{\partial z}+b_{1z}, \tag{1.13}$$

where

$$\widehat{P}=p-\frac{\rho}{2}r^2\mathbf{\Omega}^2+(2\alpha_1+\alpha_2)\left[\left(\frac{\partial u}{\partial z}\right)^2+\left(\frac{\partial v}{\partial z}\right)^2\right].$$

Boundary Layer

Boundary layer is the region near a solid where the fluid motion is affected by the solid boundary. In the bulk of the fluid the fluid is usually governed by the theory of ideal fluids. By contrast, viscosity is important in the boundary layer. The field of flow in the case of fluids of small/large viscosity can be divided , for the purpose of mathematical analysis, into two region: the thin/thick boundary near the wall, in which friction must be taken into account, and the region outside the boundary layer, where the forces due to friction are small and may be neglected.

Boundary Layer Control

Several methods of controlling the boundary layer have been developed experimentally, and also on the basis of theoretical consideration, i.e. suction and blowing, and presence of magnetic field. The purpose of these method is to affect the whole flow in the desired direction by influencing the structure of the boundary layer.

Perturbation

Perturbation theory appears in one of the oldest branches of mechanics i.e. Celestial mechanics. We know that planets exert gravitational forces on each other as a result their motion is perturbed. Perturbation method is defined as a method which yields approximate solutions for a small parameter (or a large parameter) occurring in the given equation.

Example

Consider the Cauchy problem for the parabolic equation

$$u_t + \epsilon u = u_{xx} \tag{1.14}$$

subject to the initial condition

$$u(x,0) = f(x), \quad -\infty < x < \infty. \tag{1.15}$$

Taking

$$u(x,t) = e^{-\epsilon t} v(x,t),$$

we have from Eq. (1.14)

$$v_t = v_{xx}, \tag{1.15a}$$

$$u(x,0) = f(x).$$

We are interested in finding the solution using perturbation method. For that we write

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)\epsilon^n, \tag{1.16}$$

Using Eq. (1.16) into Eq. (1.14) and (1.15), and then equating the coefficient of like power of ϵ we get

$$\frac{\partial u_0}{\partial t} - \frac{\partial^2 u_0}{\partial x^2} = 0, \tag{1.17}$$

$$\frac{\partial u_n}{\partial t} - \frac{\partial^2 u_n}{\partial x^2} = -u_{n-1}, \ n > 0 \tag{1.18}$$

with the following initial conditions

$$u_0(x,0) = f(x), \tag{1.19}$$

$$u_n(x,t) = 0, \quad n > 0. \tag{1.20}$$

Putting

$$u_0(x,t) = v(x,t), \tag{1.21}$$

where $v(x,t)$ is the solution of Eq. (1.15a) with initial condition $v(x,0) = f(x)$, we conclude that

$$u_n(x,t) = \frac{(-t)^n}{n!} v(x,t),\ n > 0 \tag{1.22}$$

By mathematical induction we have

$$\frac{\partial u_n}{\partial t} - \frac{\partial^2 u_n}{\partial x^2} = \frac{(-t)^n}{n!} \left[v_t - v_{xx}\right] - \frac{(-t)^{n-1}}{(n-1)!} v(x,t) \tag{1.23}$$

and thus the perturbation solution is

$$u(x,t) = \sum_{n=0}^{\infty} \left[\frac{(-t)^n}{n!} v(x,t)\right] \epsilon^n = \sum_{n=0}^{\infty} \frac{(-\epsilon t)^n}{n!} v(x,t), \tag{1.24}$$

or

$$u(x,t) = e^{-\epsilon t} v(x,t),$$

or

$$u(x,t) = v(x,t) - \epsilon t v(x,t) + o(\epsilon^2). \tag{1.25}$$

is the solution by using the perturbation method.

Laplace Transform

Let $f(t)$ be a given function of t then Laplace transformation is defined by

$$L\left[f(t)\right] = \overline{f}(p) = \int_0^{\infty} f(t) e^{-pt} dt,\ p > 0.$$

Exponential order

A function $f(t)$ is said to be of exponential order

$$f(t) = o\left(e^{bt}\right),$$

if there exist a real number M, a positive number b and a fixed finite positive value t_0 such that for every $t \geq t_0$

$$\lim_{t \longrightarrow \infty} |f(t)| < Me^{bt}.$$

Existence of Laplace transform

If the integral of $f(t)e^{-pt}$ exists between the limit 0 to t_0, for every finite positive t_0 and $f(t)$ is of exponential order then the Laplace transformation of $f(t)$ exists i.e.

if

$$\int_0^{t_0} f(t)e^{-pt}dt, \text{ exists}$$

and

$$f(t) = o\left(e^{bt}\right),$$

then for $p > b$, $L[f(t)]$ exists.

Derivative of a Laplace transform

Suppose $L[f(t)]$ exists and$L[f(t)] = \overline{f}(p)$, then

$$\frac{d^n}{ds^n}\overline{f}(s) = L\left[f(t)\left(-t\right)^n\right].$$

Properties of Laplace transform

If

(1)

$$L[f(t)] = \overline{f}(p),$$

then for $p > a$,

$$L\left[e^{at}f(t)\right] = f(p-a).$$

(2) Let

$$L\left[f(t)\right] = \overline{f}(p),$$

exists for $p > a$, and if 'a' is a positive constant then

$$L\left[f(t-a)u\left(t-a\right)\right] = e^{-ap}f(p),$$

$$L\left[f(t)u\left(t-a\right)\right] = e^{-ap}L\left[f(t+a)\right],$$

where $u(t-a)$ is a Unit step function.

Chapter 2

Unsteady rotating boundary layer flows of a Newtonian fluid

2.1 Introduction

In this chapter, the flow of a viscous fluid is studied between the two porous plates. Both plates and the fluid are in a state of rigid body rotation. The Laplace transform method of solution is used to find the solution. The work presented in this chapter is review of a research paper by Mukherjee and Debnath [8] . The missing details of mathematical calculations is incorporated.

2.2 The hydrodynamic problem

Let us consider the unsteady hydrodynamic flow within parallel plates at $z = 0$ and $z = h$. Initially, the whole system (fluid and porous plates) are in a rigid body rotation with constant angular velocity $\mathbf{\Omega}$ about the $z-$axis. After some instant of time, plates at $z = 0$, and $z = h$, starts oscillating with frequencies ω_1 and ω_2, respectively. The hydrodynamic equation of motion and continuity in a rotating frame are

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}.\boldsymbol{\nabla})\,\mathbf{V} + 2\mathbf{\Omega} \times \mathbf{V} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = b_1 + \frac{1}{\rho} div\mathbf{T}, \tag{2.1}$$

$$div\mathbf{V} = 0. \tag{2.2}$$

In above equation, $\mathbf{V} = (u,\ v,\ w)$ is used for the velocity vector, t for time and ρ for the density. The expression of Cauchy stress $\mathbf{T}$ for a viscous fluid is

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1, \tag{2.3}$$

Here p is the pressure, $\mathbf{I}$ is the identity tensor, μ is the coefficient of viscosity and $\mathbf{A}_1$ is first Rivlin-Ericksen tensor which is given by

$$\mathbf{A}_1 = (grad\mathbf{V}) + (grad\mathbf{V})^T. \tag{2.4}$$

We define the velocity field as

$$\mathbf{V} = [u(z,t),\ v(z,t),\ w(z,t)]. \tag{2.5}$$

Using Eq. (2.5) in Eq. (2.2) we get

$$\frac{\partial w}{\partial z} = 0, \tag{2.6}$$

which indicate that w is independent of z. Hence w is function of t or w=constant.

$$w = -W_0. \tag{2.7}$$

For uniformly porous plates where $W_0 > 0$, corresponds to the suction velocity and $W_0 < 0$, corresponds to the injection (blowing) velocity. From Eqs. (2.5) and (2.7), we can write it as

$$\mathbf{V} = [u(z,t),\ v(z,t),\ -W_0]. \tag{2.8}$$

From Eqs. (2.3), (2.4) and (2.8) we have

$$\mathbf{\Omega} \times \mathbf{V} = (-\Omega v,\ \Omega u,\ 0), \tag{2.9}$$

$$\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) = \left(-\Omega^2 x,\ -\Omega^2 y,\ 0\right), \tag{2.10}$$

$$\mathbf{L} = grad\mathbf{V} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix},$$

$$= \begin{bmatrix} 0 & 0 & \frac{\partial u}{\partial z} \\ 0 & 0 & \frac{\partial v}{\partial z} \\ 0 & 0 & 0 \end{bmatrix}, \tag{2.11}$$

$$\mathbf{L}^T = (grad\mathbf{V})^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & 0 \end{bmatrix}, \tag{2.12}$$

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T = \begin{bmatrix} 0 & 0 & \frac{\partial u}{\partial z} \\ 0 & 0 & \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & 0 \end{bmatrix}, \tag{2.13}$$

$$div\mathbf{T} = div\left(-p\mathbf{I} + \mu\mathbf{A}_1\right).$$

With the help of Eqs. (2.11)-(2.13) we can write

$$(div\mathbf{T})_x = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \upsilon\frac{\partial^2 u}{\partial z^2}, \tag{2.14}$$

$$(div\mathbf{T})_y = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \upsilon\frac{\partial^2 v}{\partial z^2}, \tag{2.15}$$

$$(div\mathbf{T})_z = -\frac{1}{\rho}\frac{\partial p}{\partial z}, \tag{2.16}$$

where υ is the kinematic viscosity. Using Eqs. (2.8) -(2.13), Eq. (2.1) in component form gives

$$\frac{\partial u}{\partial t} - W_0\frac{\partial u}{\partial z} - 2\Omega v - \Omega^2 x = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \upsilon\frac{\partial^2 u}{\partial z^2}, \tag{2.17}$$

$$\frac{\partial v}{\partial t} - W_0\frac{\partial v}{\partial z} + 2\Omega u - \Omega^2 y = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \upsilon\frac{\partial^2 v}{\partial z^2}, \tag{2.18}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z}. \tag{2.19}$$

Equation (2.19) shows that p is not a function of z. As

$$r^2 = x^2 + y^2,$$

so

$$x = \frac{1}{2}\frac{\partial}{\partial x}(r^2),\ y = \frac{1}{2}\frac{\partial}{\partial y}(r^2), \tag{2.20}$$

and thus Eqs. (2.17) and (2.18) becomes

$$\frac{\partial u}{\partial t} - W_0\frac{\partial u}{\partial z} - 2\Omega v = -\frac{\partial p^*}{\partial x} + \upsilon\frac{\partial^2 u}{\partial z^2}, \tag{2.21}$$

$$\frac{\partial v}{\partial t} - W_0\frac{\partial v}{\partial z} + 2\Omega u = -\frac{\partial p^*}{\partial y} + \upsilon\frac{\partial^2 v}{\partial z^2}, \tag{2.22}$$

where the modified pressure p^* is

$$p^* = \frac{p}{\rho} - \frac{\rho}{2}\Omega^2 r^2 \tag{2.23}$$

and Eqs. (2.21) and (2.22) can be combined in absence of pressure gradient as

$$\frac{\partial q}{\partial t} - W_0\frac{\partial q}{\partial z} + 2i\Omega q = \upsilon\frac{\partial^2 q}{\partial z^2}, \tag{2.24}$$

where $q = u + iv$.

The appropriate boundary conditions are

$$q = U^*\left(ae^{i\omega_1 t} + be^{-i\omega_1 t}\right) - U,\ on\ z = 0,\ t > 0, \tag{2.25}$$

$$q = U^*\left(ce^{i\omega_2 t} + de^{-i\omega_2 t}\right) - U,\ on\ z = h,\ t > 0, \tag{2.26}$$

$$q = 0\ for,\ 0 \le z \le h,\ t \le 0, \tag{2.27}$$

Defining the dimensionless variables,

$$\widehat{z} = \frac{zU^*}{\upsilon},\ \widehat{t} = \Omega t,\ \widehat{q} = \frac{q}{U^*},$$

initial boundary value problem is given by

$$\frac{\partial^2 q}{\partial z^2} + S\frac{\partial q}{\partial z} - iEq = \frac{E}{2}\frac{\partial q}{\partial t},\ 0 \le z \le h, t > 0, \tag{2.28}$$

$$\begin{aligned} q &= ae^{in_1 t} + be^{-in_1 t} - \frac{U}{U^*},\ on\ z = 0,\ t > 0, & (2.29)\\ q &= ce^{in_2 t} + de^{-in_2 t} - \frac{U}{U^*},\ on\ z = l,\ t > 0, & (2.30)\\ q &= 0\ for,\ 0 \le z \le l,\ t \le 0, & (2.31) \end{aligned}$$

where

$$S = \frac{W_0}{U^*},\ E = \frac{2\Omega\nu}{U^{*2}},\ l = \frac{h\upsilon}{U^{*2}},\ n_r = \frac{\omega_r}{\Omega},\ r = 1, 2.$$

2.3 Solution of the initial-boundary value problem

The initial-boundary value problem defined by Eqs. (2.28)-(2.31) is solved by Laplace transform method. For that we define the Laplace transform of $q(z,t)$ by $\overline{q}(z,p)$ as

$$\overline{q}(z,p) = \int_0^\infty e^{-pt} q(z,t)dt,\ p > 0. \tag{2.32}$$

Taking Laplace transform of Eq. (2.28) and conditions (2.29) - (2.31) we have

$$\frac{d^2\overline{q}}{dz^2} + S\frac{d\overline{q}}{dz} - (iE + \frac{E}{2}p)\overline{q} = 0, \tag{2.33}$$

$$\begin{aligned} \overline{q}(z,p) &= \frac{a}{p - in_1} + \frac{b}{p + in_1} - \frac{U}{U^*p},\ on\ z = 0, & (2.34)\\ \overline{q}(z,p) &= \frac{c}{p - in_2} + \frac{d}{p + in_2} - \frac{U}{U^*p},\ on\ z = l, & (2.35)\\ \overline{q}(z,p) &= 0\ on\ 0 \le z \le l,\ p \le 0. & (2.36) \end{aligned}$$

The solution of Eq. (2.33) is

$$\overline{q}(z,p) = e^{\alpha z}\left[A\cosh\beta z + B\sinh\beta z\right], \tag{2.37}$$

where $\alpha = -\frac{S}{2}$, $\beta = \left[\left\{\frac{(S^2+4iE)}{2E} + p\right\}\frac{E}{2}\right]^{\frac{1}{2}}$ and A and B are arbitrary constants. Using conditions (2.34) and (2.35) in Eq. (2.37) we have

$$A = \frac{a}{p - in_1} + \frac{b}{p + in_1} - \frac{U}{U^* p},$$

$$B = -\frac{\cosh \beta l}{\sinh \beta l}\left(\frac{a}{p - in_1} + \frac{b}{p + in_1} - \frac{U}{U^* p}\right) + \frac{1}{e^{\alpha l} \sinh \beta l}\left(\frac{c}{p - in_2} + \frac{d}{p + in_2} - \frac{U}{U^* p}\right),$$

Insertion of A and B in Eq. (2.37) yields

$$\begin{aligned}
\overline{q}(z,p) \quad = \quad & \frac{ae^{\alpha z}}{p - in_1}\frac{\sinh \beta(l-z)}{\sinh \beta l} + \frac{be^{\alpha z}}{p + in_1}\frac{\sinh \beta(l-z)}{\sinh \beta l} \\
& + \frac{ce^{\alpha(z-l)}}{p - in_2}\frac{\sinh \beta z}{\sinh \beta l} + \frac{de^{\alpha(z-l)}}{p + in_2}\frac{\sinh \beta z}{\sinh \beta l} \\
& - \frac{U}{U^*}\left[\frac{e^{\alpha(z-l)}}{p}\frac{\sinh \beta z}{\sinh \beta l} - \frac{e^{\alpha z}}{p}\frac{\sinh \beta(l-z)}{\sinh \beta l}\right].
\end{aligned} \tag{2.38}$$

The inverse Laplace transform of above equation will complete the solution of the problem. For that we define inverse Laplace transform as

$$q(z,t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{pt}\overline{q}(z,p)dp.$$

Substituting $\overline{q}(z,p)$ from Eq. (2.38) into above equation we can write

$$q(z,t) = \frac{1}{2\pi i}\int_{r - i\infty}^{r + i\infty} \left[\begin{array}{c} \frac{ae^{\alpha z}}{p - in_1}\frac{\sinh \beta(l-z)}{\sinh \beta l} + \frac{be^{\alpha z}}{p + in_1}\frac{\sinh \beta(l-z)}{\sinh \beta l} \\ + \frac{ce^{\alpha(z-l)}}{p - in_2}\frac{\sinh \beta z}{\sinh \beta l} + \frac{de^{\alpha(z-l)}}{p + in_2}\frac{\sinh \beta z}{\sinh \beta l} \\ - \frac{U}{U^*}\left\{\frac{e^{\alpha(z-l)}}{p}\frac{\sinh \beta z}{\sinh \beta l} - \frac{e^{\alpha z}}{p}\frac{\sinh \beta(l-z)}{\sinh \beta l}\right\} \end{array}\right] e^{pt}dp,$$

or

$$q(z,t) = I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \tag{2.39}$$

where

$$I_1 = \frac{1}{2\pi i}\int_{r - i\infty}^{r + i\infty} \frac{ae^{\alpha z}}{p - in_1}\frac{\sinh \beta(l-z)}{\sinh \beta l}e^{pt}dp,$$

$$I_2 = \frac{1}{2\pi i}\int_{r - i\infty}^{r + i\infty} \frac{be^{\alpha z}}{p + in_1}\frac{\sinh \beta(l-z)}{\sinh \beta l}e^{pt}dp,$$

$$I_3 = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{ce^{\alpha(z-l)}}{p-in_2}\frac{\sinh\beta z}{\sinh\beta l}e^{pt}dp,$$

$$I_4 = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{de^{\alpha(z-l)}}{p+in_2}\frac{\sinh\beta z}{\sinh\beta l}e^{pt}dp,$$

$$I_5 = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} -\frac{U}{U^*}\frac{e^{\alpha(z-l)}}{p}\frac{\sinh\beta z}{\sinh\beta l}e^{pt}dp,$$

$$I_6 = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{U}{U^*}\frac{e^{\alpha z}}{p}\frac{\sinh\beta(l-z)}{\sinh\beta l}e^{pt}dp.$$

Now we are interested in the solution of the integrals using complex variable theory. We consider

$$I_1 = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{ae^{\alpha z}}{p-in_1}\frac{\sinh\beta(l-z)}{\sinh\beta l}e^{pt}dp. \tag{2.40}$$

Here the integrand has a simple pole at $p = in_1$and the value of p for which $sinh\beta l = 0$, i.e., $\beta l = \frac{n\pi}{i}$. The residue at $p = in_1$, is $a_1$which is

$$a_1 = ae^{\alpha z+in_1 t}\frac{\sinh\lambda_1(l-z)}{\sinh\lambda_1 l} \tag{2.41}$$

and a_2 is the residue at the pole when $sinh\beta l = 0$. By expansion formula

$$L^{-1}\left\{\frac{L(p)}{Q(p)}\right\} = \sum_{r=1}^{n}\frac{L(\alpha_r)}{Q^{'}(\alpha_r)}e^{\alpha_r t}, \tag{2.42}$$

where $Q(p)$has n distinct zeros α_r we have

$$a_2 = -\sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r a e^{\alpha z+\lambda t}\sin r\pi\left\{(1-\frac{z}{l})\right\}}{El^2\left(\frac{S^2+4iE}{2E}+\frac{2r^2\pi^2}{El^2}+in_1\right)} \tag{2.43}$$

and thus the integral I_1 gives

$$I_1 = ae^{\alpha z+in_1 t}\frac{\sinh\lambda_1(l-z)}{\sinh\lambda_1 l} - \sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r a e^{\alpha z+\lambda t}\sin r\pi\left\{(1-\frac{z}{l})\right\}}{El^2\left(\frac{S^2+4iE}{2E}+\frac{2r^2\pi^2}{El^2}+in_1\right)}. \tag{2.44}$$

On a similar pattern

$$I_2 = be^{\alpha z - in_1 t}\frac{\sinh \lambda_1'(l-z)}{\sinh \lambda_1' l} - \sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r b e^{\alpha z-\lambda t}\sin r\pi\left\{(1-\frac{z}{l})\right\}}{El^2\left(\frac{S^2+4iE}{2E}+\frac{2r^2\pi^2}{El^2}-in_1\right)}, \tag{2.45}$$

$$I_3 = ce^{\alpha(z-l)+in_2 t}\frac{\sinh \lambda_2 z}{\sinh \lambda_2 l} - \sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r c e^{\alpha(z-l)-\lambda t}\sin(\frac{r\pi z}{l})}{El^2\left(\frac{S^2+4iE}{2E}+\frac{2r^2\pi^2}{El^2}+in_1\right)}, \tag{2.46}$$

$$I_4 = de^{\alpha(z-l)-in_2 t}\frac{\sinh \lambda_2' z}{\sinh \lambda_2' l} - \sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r d e^{\alpha(z-l)-\lambda t}\sin(\frac{r\pi z}{l})}{El^2\left(\frac{S^2+4iE}{2E}+\frac{2r^2\pi^2}{El^2}-in_1\right)}, \tag{2.47}$$

$$I_5 = -\frac{U}{U^*}e^{\alpha(z-l)}\frac{\sinh \lambda_3 z}{\sinh \lambda_3 l} - \frac{U}{U^*}\sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r e^{\alpha(z-l)-\lambda t}\sin(\frac{r\pi z}{l})}{2r^2\pi^2+\frac{l^2}{2}(S^2+4iE)}, \tag{2.48}$$

$$I_6 = -\frac{U}{U^*}e^{\alpha z}\frac{\sinh \lambda_3(l-z)}{\sinh \lambda_3 l} - \frac{U}{U^*}\sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r e^{(\alpha z-\lambda t)}\sin r\pi\left\{(1-\frac{z}{l})\right\}}{2r^2\pi^2+\frac{l^2}{2}(S^2+4iE)}. \tag{2.49}$$

Using the above values of integrals, Eq. (2.39) gives

$$
\begin{aligned}
q(z,t) &= ae^{\alpha z+in_1 t}\frac{\sinh\lambda_1(l-z)}{\sinh\lambda_1 l} \\
&\quad +be^{\alpha z-in_1 t}\frac{\sinh\lambda_1'(l-z)}{\sinh\lambda_1' l} \\
&\quad +ce^{\alpha(z-l)+in_2 t}\frac{\sinh\lambda_2 z}{\sinh\lambda_2 l} \\
&\quad +de^{\alpha(z-l)-in_2 t}\frac{\sinh\lambda_2^{'} z}{\sinh\lambda_2^{'} l} \\
&\quad -\frac{U}{U^*}\left[e^{\alpha(z-l)}\frac{\sinh\lambda_3 z}{\sinh\lambda_3 l}+e^{\alpha z}\frac{\sinh\lambda_3(l-z)}{\sinh\lambda_3 l}\right] \\
&\quad -\sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r a e^{\alpha z+\lambda t}\sin r\pi\left\{(1-\frac{z}{l})\right\}}{El^2\left(\frac{S^2+4iE}{2E}+\frac{2r^2\pi^2}{El^2}+in_1\right)} \\
&\quad -\sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r b e^{\alpha z-\lambda t}\sin r\pi\left\{(1-\frac{z}{l})\right\}}{El^2\left(\frac{S^2+4iE}{2E}+\frac{2r^2\pi^2}{El^2}-in_1\right)} \\
&\quad -\sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r c e^{\alpha(z-l)-\lambda t}\sin(\frac{r\pi z}{l})}{El^2\left(\frac{S^2+4iE}{2E}+\frac{2r^2\pi^2}{El^2}+in_1\right)} \\
&\quad -\sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r d e^{\alpha(z-l)-\lambda t}\sin(\frac{r\pi z}{l})}{-El^2\left(\frac{S^2+4iE}{2E}+\frac{2r^2\pi^2}{El^2}-in_1\right)} \\
&\quad -\frac{U}{U^*}\sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r e^{\alpha(z-l)-\lambda t}\sin(\frac{r\pi z}{l})}{2r^2\pi^2+\frac{l^2}{2}\left(S^2+4iE\right)} \\
&\quad -\frac{U}{U^*}\sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r e^{(\alpha z-\lambda t)}\sin r\pi\left\{(1-\frac{z}{l})\right\}}{2r^2\pi^2+\frac{l^2}{2}\left(S^2+4iE\right)},
\end{aligned}
\tag{2.50}
$$

in which

$$
\begin{aligned}
\lambda_r &= \frac{1}{2}\left[S^2+2iE(2+n_r)\right]^{\frac{1}{2}},\ r=1,2, \\
\lambda_r' &= \frac{1}{2}\left[S^2+2iE(2-n_r)\right]^{\frac{1}{2}},\ r=1,2, \\
\lambda_3 &= \frac{1}{2}\left[S^2+4iE\right]^{\frac{1}{2}}, \\
\lambda &= \left(\frac{S^2+4iE}{2E}+\frac{2r^2\pi^2}{El^2}\right).
\end{aligned}
$$

For $S \neq 0$, it is noted that

$$\lim_{t\to\infty}\lim_{n_r\to 2} q(z,t) = \lim_{n_r\to 2}\lim_{t\to\infty} q(z,t).$$

The steady state solution consists of multiple boundary layers of thickness of the orders in case of suction as

$$\begin{aligned}
&\frac{\upsilon}{U^*\left(\frac{S}{2}+a_r\right)},\ (r = 1,3),\ \frac{\upsilon}{U^*\left(\frac{S}{2}+a_2'\right)},\ on\ z=0;\\
&\frac{\upsilon}{U^*\left(\frac{S_1}{2}-a_r\right)},\ (r = 2,3),\ \frac{\upsilon}{U^*\left(\frac{S_1}{2}-a_2'\right)},\ on\ z=l,
\end{aligned}$$

where

$$\begin{aligned}
a_r &= \frac{1}{2\sqrt{2}}\left[\left\{S^4+4E^2(2+n_r)^2\right\}^{\frac{1}{2}}+S^2\right]^{\frac{1}{2}},\\
b_r &= \frac{1}{2\sqrt{2}}\left[\left\{S^4+4E^2(2+n_r)^2\right\}^{\frac{1}{2}}-S^2\right]^{\frac{1}{2}},\\
a_r' &= \frac{1}{2\sqrt{2}}\left[\left\{S^4+4E^2(2-n_r)^2\right\}^{\frac{1}{2}}+S^2\right]^{\frac{1}{2}},\\
b_r' &= \frac{1}{2\sqrt{2}}\left[\left\{S^4+4E^2(2-n_r)^2\right\}^{\frac{1}{2}}-S^2\right]^{\frac{1}{2}},\\
a_3 &= \frac{1}{2\sqrt{2}}\left[\left(S^4+16E^2\right)^{\frac{1}{2}}+S^2\right]^{\frac{1}{2}},\\
b_3 &= \frac{1}{2\sqrt{2}}\left[\left(S^4+16E^2\right)^{\frac{1}{2}}-S^2\right]^{\frac{1}{2}}.
\end{aligned}$$

Now $S=-S_1$, $S_1>0$, corresponds to injection on the plate at $z=l$, $\lambda_r = a_r + ib_r$, $\lambda_r' = a_r' + ib_r'$, $(r=1,2)$ and $\lambda_3 = a_3 + ib_3$. For $\omega_2 = 2\Omega$ $(n_2=2)$, it is found that boundary condition when $l\to\infty$ is not satisfied and this difficulty has been removed by considering the hydromagnetic problem in the next section.

2.4 The hydromagnetic problem

This section deals the problem considered in the section (1.3) in the presence of a constant applied magnetic field $\mathbf{B}_0$ parallel to the z-axis. The equation of motion now is

$$\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}.\boldsymbol{\nabla})\mathbf{V} + 2\boldsymbol{\Omega} \times \mathbf{V} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = -\frac{1}{\rho}\boldsymbol{\nabla} p + \frac{1}{\rho}\mathbf{J} \times \mathbf{B} + \upsilon \boldsymbol{\nabla}^2 \mathbf{V}, \tag{2.52}$$

where $\frac{1}{\rho}\mathbf{J} \times \mathbf{B}$, is the Lorentz force. $\mathbf{B}$ is total magnetic field so that $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$, $\mathbf{J}$ is the current density, $\mathbf{b}$ is the induced magnetic field and by Ohm's Law

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{V} \times \mathbf{B}).$$

The Maxwell's equations are

$$div\mathbf{B} = 0, \tag{2.53a}$$

$$curl\mathbf{B} = \mu_1 \mathbf{J}, \tag{2.53b}$$

$$curl\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2.53c}$$

where μ_1 is the magnetic permeability, $\mathbf{E}$ is the electric field and σ is the electrically conductivity of the fluid. For small magnetic Reynold's number, the induced magnetic field is neglected and thus

$$\begin{aligned} \frac{1}{\rho}\mathbf{J} \times \mathbf{B} &= n^* \mathbf{V}, \\ n^* &= -\frac{\sigma}{\rho}\mathbf{B}_0^2. \end{aligned}$$

The non dimensional boundary layer equation in the hydromagnetic case is given by

$$\frac{\partial^2 q}{\partial z^2} + S\frac{\partial q}{\partial z} - (n + iE)q = \frac{E}{2}\frac{\partial q}{\partial t},\ 0 \le z \le l,\ t > 0, \tag{2.54}$$

with the following boundary and initial conditions

$$\begin{aligned} q(z,t) &= ae^{in_1 t} + be^{-in_1 t} - \frac{U}{U^*},\ \text{on } z = 0,\ t > 0, \\ q(z,t) &= ce^{in_2 t} + de^{-in_2 t} - \frac{U}{U^*},\ \text{on } z = l,\ t > 0, \\ q(z,t) &= 0,\ 0 \le z \le l,\ t \le 0 \end{aligned}$$

where $n = \frac{n^* \upsilon}{U^{*2}}$, $n_r = \frac{\omega_r}{\Omega}, r = 1, 2$.

The Eq. (2.54) together with the above conditions can be solved by Laplace transform method. Thus using the similar procedure as in section (1.3) we have

$$q(z,t) = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \begin{bmatrix} \frac{ae^{\alpha z}}{p-in_1}\frac{\sinh\eta(l-z)}{\sinh\eta l} + \frac{be^{\alpha z}}{p+in_1}\frac{\sinh\eta(l-z)}{\sinh\eta l} \\ + \frac{ce^{\alpha(z-l)}}{p-in_2}\frac{\sinh\eta z}{\sinh\eta l} + \frac{de^{\alpha(z-l)}}{p+in_2}\frac{\sinh\eta z}{\sinh\eta l} \\ -\frac{U}{U^*}\left\{\frac{e^{\alpha(z-l)}}{p}\frac{\sinh\eta z}{\sinh\eta l} - \frac{e^{\alpha z}}{p}\frac{\sinh\eta(l-z)}{\sinh\eta l}\right\} \end{bmatrix} e^{pt}dp,$$

or

$$q(z,t) = I_1^* + I_2^* + I_3^* + I_4^* + I_5^* + I_6^*, \tag{2.55}$$

where

$$I_1^* = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{ae^{\alpha z}}{p-in_1}\frac{\sinh\eta(l-z)}{\sinh\eta l}e^{pt}dp, \tag{2.56}$$

$$I_2^* = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{be^{\alpha z}}{p+in_1}\frac{\sinh\eta(l-z)}{\sinh\eta l}e^{pt}dp, \tag{2.57}$$

$$I_3^* = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{ce^{\alpha(z-l)}}{p-in_2}\frac{\sinh\eta z}{\sinh\eta l}e^{pt}dp, \tag{2.58}$$

$$I_4^* = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{de^{\alpha(z-l)}}{p+in_2}\frac{\sinh\eta z}{\sinh\eta l}e^{pt}dp, \tag{2.59}$$

$$I_5^* = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} -\frac{U}{U^*}\frac{e^{\alpha(z-l)}}{p}\frac{\sinh\eta z}{\sinh\eta l}e^{pt}dp, \tag{2.60}$$

$$I_6^* = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{U}{U^*}\frac{e^{\alpha z}}{p}\frac{\sinh\eta(l-z)}{\sinh\eta l}e^{pt}dp \tag{2.61}$$

where

$$\alpha = -\frac{S}{2}, \eta = \left[\left\{\frac{(S^2 + 4iE + 4n)}{2E} + p\right\}\frac{E}{2}\right]^{\frac{1}{2}},$$

Consider

$$I_1^* = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{ae^{\alpha z}}{p-in_1}\frac{\sinh\eta(l-z)}{\sinh\eta l}e^{pt}dp. \tag{2.62}$$

In order to evaluate the integral appearing in above equation we use residue theorem.. The integrand has a simple pole at $p = in_1$. This implies $\eta = \mu_1$, and another simple pole occurs where $sinh\eta l = 0$, i.e. $\eta l = \frac{n\pi}{i}$. Residue at $p = in_1$, is $\widetilde{a}_1$ which is

$$\widetilde{a}_1 = ae^{\alpha z + in_1 t}\frac{\sinh\mu_1(l-z)}{\sinh\mu_1 l}, \tag{2.63}$$

and $\widetilde{a}_2$ is the residue at the pole when $sinh\eta l = 0$. This implies that

$p = -\delta = -(\frac{2r^2\pi^2}{El^2} + \frac{s^2+4iE+4n}{2E})$. By expansion formula

$$L^{-1}\left\{\frac{L(p)}{Q(p)}\right\} = \sum_{r=1}^{n}\frac{L(\alpha_r)}{Q^{'}(\alpha_r)}e^{\alpha_r t}, \tag{2.64}$$

where $Q(p)$ has n distinct zeros α_r. The residue is

$$\widetilde{a}_2 = -\sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r a e^{\alpha z-\delta t}\sin r\pi\left\{(1-\frac{z}{l})\right\}}{El^2\left(\frac{s^2+4iE+4n}{2E}+\frac{2r^2\pi^2}{El^2}+in_1\right)}, \tag{2.65}$$

and the integral I_1^* gives

$$I_1^* = ae^{\alpha z+in_1 t}\frac{\sinh\mu_1(l-z)}{\sinh\mu_1 l} - \sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r a e^{\alpha z-\delta t}\sin r\pi\left\{(1-\frac{z}{l})\right\}}{El^2\left(\frac{s^2+4iE+4n}{2E}+\frac{2r^2\pi^2}{El^2}+in_1\right)}. \tag{2.66}$$

Similarly evaluation of the integrals in Eq. (2.55) has been made using the residue theorem and are given by

$$I_2^* = be^{\alpha z-in_1 t}\frac{\sinh\mu_1'(l-z)}{\sinh\mu_1' l} - \sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r b e^{\alpha z-\delta t}\sin r\pi\left\{(1-\frac{z}{l})\right\}}{El^2\left(\frac{S^2+4iE+4n}{2E}+\frac{2r^2\pi^2}{El^2}-in_1\right)}, \tag{2.67}$$

$$I_3^* = ce^{\alpha(z-l)+in_2 t}\frac{\sinh\mu_2 z}{\sinh\mu_2 l} - \sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r c e^{\alpha(z-l)-\delta t}\sin(\frac{r\pi z}{l})}{El^2\left(\frac{S^2+4iE+4n}{2E}+\frac{2r^2\pi^2}{El^2}+in_1\right)}, \tag{2.74}$$

$$I_4^* = de^{\alpha(z-l)-in_2 t}\frac{\sinh\mu_2^{'} z}{\sinh\mu_2^{'} l} - \sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r d e^{\alpha(z-l)-\delta t}\sin(\frac{r\pi z}{l})}{El^2\left(\frac{S^2+4iE+4n}{2E}+\frac{2r^2\pi^2}{El^2}-in_1\right)}, \tag{2.75}$$

$$I_5^* = -\frac{u}{u^*}e^{\alpha(z-l)}\frac{\sinh\mu_3 z}{\sinh\mu_3 l} - \frac{U}{U^*}\sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r e^{\alpha(z-l)-\delta t}\sin(\frac{r\pi z}{l})}{2r^2\pi^2+\frac{l^2}{2}\left(S^2+4iE+4n\right)}, \tag{2.76}$$

$$I_6^* = -\frac{u}{u^*}e^{\alpha z}\frac{\sinh\mu_3(l-z)}{\sinh\mu_3 l} - \frac{U}{U^*}\sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r e^{(\alpha z-\delta t)}\sin r\pi\left\{(1-\frac{z}{l})\right\}}{2r^2\pi^2+\frac{l^2}{2}\left(S^2+4iE+4n\right)}. \tag{2.77}$$

Using the above values of integrals, Eq. (2.55) yields

$$\begin{aligned}
q(z,t) \;=\;& ae^{\alpha z+in_1 t}\frac{\sinh\mu_1(l-z)}{\sinh\mu_1 l}\\
&+be^{\alpha z-in_1 t}\frac{\sinh\mu_1'(l-z)}{\sinh\mu_1' l}\\
&+ce^{\alpha(z-l)+in_2 t}\frac{\sinh\mu_2 z}{\sinh\mu_2 l}\\
&+de^{\alpha(z-l)-in_2 t}\frac{\sinh\mu_2^{'} z}{\sinh\mu_2^{'} l}\\
&-\frac{U}{U^*}\left[e^{\alpha(z-l)}\frac{\sinh\mu_3 z}{\sinh\mu_3 l}+e^{\alpha z}\frac{\sinh\mu_3(l-z)}{\sinh\mu_3 l}\right]\\
&-\sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r a e^{\alpha z-\delta t}\sin r\pi\left\{(1-\frac{z}{l})\right\}}{El^2\left(\frac{S^2+4iE+4n}{2E}+\frac{2r^2\pi^2}{El^2}+in_1\right)}\\
&-\sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r b e^{\alpha z-\delta t}\sin r\pi\left\{(1-\frac{z}{l})\right\}}{El^2\left(\frac{S^2+4iE+4n}{2E}+\frac{2r^2\pi^2}{El^2}-in_1\right)}\\
&-\sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r c e^{\alpha(z-l)-\delta t}\sin(\frac{r\pi z}{l})}{El^2\left(\frac{S^2+4iE+4n}{2E}+\frac{2r^2\pi^2}{El^2}+in_1\right)}\\
&-\sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r d e^{\alpha(z-l)-\lambda t}\sin(\frac{r\pi z}{l})}{El^2\left(\frac{S^2+4iE+4n}{2E}+\frac{2r^2\pi^2}{El^2}-in_1\right)}\\
&-\frac{U}{U^*}\sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r e^{\alpha(z-l)-\delta t}\sin(\frac{r\pi z}{l})}{2r^2\pi^2+\frac{l^2}{2}\left(S^2+4iE+4n\right)}\\
&-\frac{U}{U^*}\sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r e^{(\alpha z-\delta t)}\sin r\pi\left\{(1-\frac{z}{l})\right\}}{2r^2\pi^2+\frac{l^2}{2}\left(S^2+4iE+4n\right)}
\end{aligned} \tag{2.78}$$

where

$$\begin{aligned}
\mu_r &= \frac{1}{2}\left[s^2+4n+2iE(2+n_r)\right]^{\frac{1}{2}}, r=1,2,\\
\mu_r' &= \frac{1}{2}\left[s^2+4n+2iE(2-n_r)\right]^{\frac{1}{2}}, r=1,2,\\
\widetilde{\lambda}_3 &= \frac{1}{2}\left[s^2+4iE+4n\right]^{\frac{1}{2}},\\
\delta &= \left(\frac{s^2+4iE+4n}{2E}+\frac{2r^2\pi^2}{El^2}\right).
\end{aligned}$$

We note that Eq. (2.78) gives the unsteady rotating flow of a viscous fluid between the two porous plates. The flow is strongly dependent upon suction and blowing velocities. The unsteady flow (2.78) is sum of the steady and transient parts. The boundary layer thicknesses in case of suction for this flow are of the order of

$$\begin{aligned}
&\frac{\upsilon}{U^*\left(\frac{S}{2}+c_r\right)},\ (r = 1,3),\ \frac{\upsilon}{U^*\left(\frac{S}{2}+c_2'\right)},\ on\ z=0,\\
&\frac{\upsilon}{U^*\left(\frac{S_1}{2}-c_r\right)},\ (r = 2,3),\ \frac{\upsilon}{U^*\left(\frac{S_1}{2}-c_2'\right)},\ on\ z=l,
\end{aligned}$$

$\mu_r = c_r + id_r,\ \mu_r^{'} = c_r' + id_r',\ (r=1,2)\ \mu_3 = c_3 + id_3$, with

$$\begin{aligned}
c_r &= \frac{1}{2\sqrt{2}}\left[\left\{(S^2+4n)^2+4E^2(2+n_r)^2\right\}^{\frac{1}{2}}+(S^2+4n)\right]^{\frac{1}{2}},\\
d_r &= \frac{1}{2\sqrt{2}}\left[\left\{(S^2+4n)^2+4E^2(2+n_r)^2\right\}^{\frac{1}{2}}-(S^2+4n)\right]^{\frac{1}{2}},\\
c_r' &= \frac{1}{2\sqrt{2}}\left[\left\{(S^2+4n)^2+4E^2(2-n_r)^2\right\}^{\frac{1}{2}}+(S^2+4n)\right]^{\frac{1}{2}},\\
d_r' &= \frac{1}{2\sqrt{2}}\left[\left\{(S^2+4n)^2+4E^2(2-n_r)^2\right\}^{\frac{1}{2}}-(S^2+4n)\right]^{\frac{1}{2}},\\
c_3 &= \frac{1}{2\sqrt{2}}\left[\left\{(S^2+4n)^2+16E^2\right\}^{\frac{1}{2}}+(S^2+4n)\right]^{\frac{1}{2}},\\
d_3 &= \frac{1}{2\sqrt{2}}\left[\left\{(S^2+4n)+16E^2\right\}^{\frac{1}{2}}-(S^2+4n)\right]^{\frac{1}{2}},
\end{aligned}$$

It is noted that expression (2.78) is meaningful solution valid for all values of the frequencies including the resonant frequency. The solution (2.78) satisfies the boundary condition for

resonant case ($\omega = 2\Omega$) which is distinct feature from the hydrodynamic problem when $l \to \infty$.

Chapter 3

Unsteady flow of a second grade fluid between two plates

3.1 Introduction

An analytic solution of the time dependent equation is obtained for the flow between two porous plates executing oscillations with different frequencies. The fluid considered is second grade and electrically conducting. A pressure gradient of the oscillating type is also imposed. The obtained analytical solution described the flow at large and small times. The solutions depend on the suction and injection velocities, frequencies, magnetic parameter and material constant of a second grade fluid.

3.2 Problem formulation

An incompressible second grade fluid filling the space between two porous plates is considered. The z-axis is perpendicular to the plates and the upper and lower plates are given at $z = 0$, and $z = h$, respectively. Both plates and fluid are in a state of rigid body rotation. The flow is induced due to oscillations of the plates and imposed oscillating pressure gradient. The governing boundary layer equations can be written from Eqs (1.11) and (1.12) as

$$\frac{\partial q}{\partial t} - W_0 \frac{\partial q}{\partial z} + (2\Omega i + n)q = \upsilon \frac{\partial^2 q}{\partial z^2} + \alpha(\frac{\partial^3 q}{\partial z^2 \partial t} - W_0 \frac{\partial^3 q}{\partial z^3}) + a_1 e^{int} + b_1 e^{-int} \qquad (3.1)$$

with the following initial and boundary conditions

$$\begin{aligned} q &= ae^{in_1 t} + be^{-in_1 t} - \frac{U}{U^*},\ on\ z = 0,\ t > 0, \\ q &= ce^{in_2 t} + de^{-in_2 t} - \frac{U}{U^*},\ on\ z = l,\ t > 0, \\ q &= ce^{in_2 t} + de^{-in_2 t} - \frac{U}{U^*},\ on\ z = l,\ t > 0, \end{aligned} \tag{3.2}$$

where $q = u + iv$. Introducing the dimensionless parameters

$$\widehat{z} = \frac{zU^*}{\upsilon},\ \widehat{t} = \Omega t,\ \widehat{q} = \frac{q}{U^*},$$

equation (3.1) and conditions (3.2) become of the following form

$$\frac{\partial^2 q}{\partial z^2} + \alpha_1(\frac{\partial^3 q}{\partial z^2 \partial t} - \frac{\upsilon_1}{\upsilon_2} S \frac{\partial^3 q}{\partial z^3}) + S\frac{\partial q}{\partial z} = \frac{E}{2}\frac{\partial q}{\partial t} + iEq + nq + a_1 e^{int} + b_1 e^{-int}, \tag{3.3}$$

$$q = ae^{in_1 t} + be^{-in_1 t} - \frac{U}{U^*},\ on\ z = 0,\ t > 0, \tag{3.4a}$$

$$q = ce^{in_2 t} + de^{-in_2 t} - \frac{U}{U^*},\ on\ z = l,\ t > 0, \tag{3.4b}$$

$$q = ce^{in_2 t} + de^{-in_2 t} - \frac{U}{U^*},\ on\ z = l,\ t > 0, \tag{3.4c}$$

where

$$\begin{aligned} E &= \frac{2\Omega\upsilon}{U^{*2}},\ n_r = \frac{\omega_r}{\Omega},\ r = 1, 2.\ S = \frac{W_0}{U^*}, \\ \upsilon_1 &= \frac{\Omega}{\upsilon},\ \upsilon_2 = \frac{U^{*2}}{\upsilon^2},\ \alpha_1 = \alpha\upsilon_1,\ l = \frac{h\upsilon}{U^*}. \end{aligned}$$

3.3 Analytical solution

Let us define the Laplace transform of $q(z,t)$ as

$$\overline{q}(z,p) = \int_0^\infty e^{-pt} q(z,t)dt,\ p > 0.$$

With the use of above definition, the governing problem in p-plane becomes

$$(1+p\alpha_1)\frac{\partial^2\overline{q}}{\partial z^2}-\alpha_1\frac{v_1}{v_2}S\frac{\partial^3\overline{q}}{\partial z^3}+S\frac{\partial\overline{q}}{\partial z}=\frac{E}{2}p\overline{q}+iE\overline{q}+n\overline{q}+\frac{a_1}{p-in}+\frac{b_1}{p+in}, \tag{3.5}$$

$$\begin{aligned}
\overline{q}(z,p) &= \frac{a}{p-in_1}+\frac{b}{p+in_1}-\frac{U}{U^*p},\ on\ z=0, &(3.5a)\\
\overline{q}(z,p) &= \frac{c}{p-in_2}+\frac{d}{p+in_2}-\frac{U}{U^*p},\ on\ z=l, &(3.5b)\\
\overline{q}(z,p) &= 0\ on\ 0\le z\le l,\ p\le 0. &(3.5c)
\end{aligned}$$

We are interested in finding a perturbation solution taking α as a small parameter and thus write

$$\overline{q}=\overline{q}_0+\alpha\overline{q}_1+o(\alpha^2). \tag{3.5d}$$

Making use of Eq. (3.5d) into Eq (3.5) and conditions (3.5a)-(3.5c) and then equating the coefficients of like powers of α we obtain

Zeroth order system

$$\frac{d^2\overline{q}_0}{dz^2}+S\frac{d\overline{q}_0}{dz}-(n+iE+\frac{E}{2}p)\overline{q}_0=\frac{a_1}{p-in}+\frac{b_1}{p+in} \tag{3.6}$$

$$\begin{aligned}
\overline{q}_0(z,p) &= \frac{a}{p-in_1}+\frac{b}{p+in_1}-\frac{U}{U^*p},\ on\ z=0, &(3.6a)\\
\overline{q}_0(z,p) &= \frac{c}{p-in_2}+\frac{d}{p+in_2}-\frac{U}{U^*p},\ on\ z=l, &(3.6b)\\
\overline{q}_0(z,p) &= 0\ on\ 0\le z\le l,\ p\le 0. &(3.6c)
\end{aligned}$$

First order system

$$\frac{d^2\overline{q}_1}{dz^2}+S\frac{d\overline{q}_1}{dz}-(n+iE+\frac{E}{2}p)\overline{q}_1=-p\frac{d^2\overline{q}_0}{dz^2}+\frac{v_1}{v_2}S\frac{d^3\overline{q}_0}{dz^3}, \tag{3.7}$$

$$\overline{q}_1(z,p) = 0, \ on \ z=0, \tag{3.7a}$$

$$\overline{q}_1(z,p) = 0, \ on \ z=l, \tag{3.7b}$$

$$\overline{q}_1(z,p) = 0, \ on \ 0 \le z \le l, \ p \le 0. \tag{3.7c}$$

We note that zeroth order problem is identical to the problem discussed in chapter 2. In order to avoid repetition we only write the solution directly which is

$$\begin{aligned}
\overline{q}_0(z,p) = & \frac{ae^{\alpha z}}{p-in_1}\frac{\sinh\eta(l-z)}{\sinh\eta l} + \frac{be^{\alpha z}}{p+in_1}\frac{\sinh\eta(l-z)}{\sinh\eta l} \\
& + \frac{ce^{\alpha(z-l)}}{p-in_2}\frac{\sinh\eta z}{\sinh\eta l} + \frac{de^{\alpha(z-l)}}{p+in_2}\frac{\sinh\eta z}{\sinh\eta l} \\
& - \frac{U}{U^*}\left[\frac{e^{\alpha(z-l)}}{p}\frac{\sinh\eta z}{\sinh\eta l} - \frac{e^{\alpha z}}{p}\frac{\sinh\eta(l-z)}{\sinh\eta l}\right] \\
& + \frac{1}{k}\left(\frac{a_1e^{\alpha z}}{p-in}\right)\frac{\sinh\eta(l-z)}{\sinh\eta l} + \frac{1}{k}\left(\frac{b_1e^{\alpha z}}{p+in}\right)\frac{\sinh\eta(l-z)}{\sinh\eta l} \\
& + \frac{1}{k}\left(\frac{a_1e^{\alpha(z-l)}}{p-in}\right)\frac{\sinh\eta z}{\sinh\eta l} + \frac{1}{k}\left(\frac{b_1e^{\alpha(z-l)}}{p+in}\right)\frac{\sinh\eta z}{\sinh\eta l} \\
& - \frac{1}{k}\left(\frac{a_1}{p-in} + \frac{b_1}{p+in}\right).
\end{aligned} \tag{3.8}$$

For finding q_0 we have to invert $\overline{q}_0$ and thus define

$$q_0(z,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{pt}\overline{q}_0(z,p)dp. \tag{3.9}$$

Substituting Eq. (3.8) into Eq. (3.9) and then using the procedure of chapter 2 for the evaluation

of integrals we arrive at

$$
\begin{aligned}
q_0(z,t) \;=\;& a e^{\alpha z + i n_1 t}\frac{\sinh\mu_1(l-z)}{\sinh\mu_1 l} + b e^{\alpha z - i n_1 t}\frac{\sinh\mu_1'(l-z)}{\sinh\mu_1' l} \\
&+ c e^{\alpha(z-l)+i n_2 t}\frac{\sinh\mu_2 z}{\sinh\mu_2 l} + d e^{\alpha(z-l)-i n_2 t}\frac{\sinh\mu_2^{'} z}{\sinh\mu_2^{'} l} \\
&+ a_1 e^{\alpha z + i n t}\frac{\sinh\mu_4(l-z)}{k_1\sinh\mu_4 l} - \frac{a_1 e^{\alpha z-\lambda t}}{\lambda + i n}\frac{\sinh\alpha(l-z)}{\sinh\alpha l} \\
&+ b_1 e^{\alpha z - i n t}\frac{\sinh\mu_4'(l-z)}{k_1'\sinh\mu_4' l} - \frac{b_1 e^{\alpha z - i n t}}{\lambda - i n}\frac{\sinh\alpha(l-z)}{\sinh\alpha l} \\
&+ a_1 e^{\alpha(z-l)+i n t}\frac{\sinh\mu_4 z}{k_1\sinh\mu_4 l} - \frac{a_1 e^{(\alpha z-\lambda t)+i n t}}{\lambda + i n}\frac{\sinh\alpha z}{\sinh\alpha l} \\
&+ b_1 e^{\alpha(z-l)- i n t}\frac{\sinh\mu_4^{'} z}{k_1'\sinh\mu_4^{'} l} - \frac{b_1 e^{(\alpha z - i n t)- i n t}}{\lambda - i n}\frac{\sinh\alpha z}{\sinh\alpha l} \\
&- \frac{a_1 e^{-\lambda t}}{\lambda + i n} + \frac{a_1 e^{i n t}}{k_1} - \frac{b_1 e^{-\lambda t}}{\lambda - i n} + \frac{b_1 e^{i n t}}{k_1'} \\
&- \frac{U}{U^*}\left[e^{\alpha(z-l)}\frac{\sinh\mu_3 z}{\sinh\mu_3 l} + e^{\alpha z}\frac{\sinh\mu_3(l-z)}{\sinh\mu_3 l}\right] \\
&- \sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r a e^{\alpha z-\delta t}\sin r\pi\left\{(1-\frac{z}{l})\right\}}{El^2\left(\frac{S^2+4iE+4n}{2E} + \frac{2r^2\pi^2}{El^2} + i n_1\right)} \\
&- \sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r b e^{\alpha z-\delta t}\sin r\pi\left\{(1-\frac{z}{l})\right\}}{El^2\left(\frac{S^2+4iE+4n}{2E} + \frac{2r^2\pi^2}{El^2} - i n_1\right)} \\
&- \sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r c e^{\alpha(z-l)-\delta t}\sin(\frac{r\pi z}{l})}{El^2\left(\frac{S^2+4iE+4n}{2E} + \frac{2r^2\pi^2}{El^2} + i n_1\right)} \\
&- \sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r d e^{\alpha(z-l)-\lambda t}\sin(\frac{r\pi z}{l})}{El^2\left(\frac{S^2+4iE+4n}{2E} + \frac{2r^2\pi^2}{El^2} - i n_1\right)} \\
&- \sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r a_1 e^{\alpha z-\delta t}\sin r\pi\left\{(1-\frac{z}{l})\right\}}{k_2 El^2\left(\frac{S^2+4iE+4n}{2E} + \frac{2r^2\pi^2}{El^2} + i n\right)} \\
&- \sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r b_1 e^{\alpha z-\delta t}\sin r\pi\left\{(1-\frac{z}{l})\right\}}{k_2 El^2\left(\frac{S^2+4iE+4n}{2E} + \frac{2r^2\pi^2}{El^2} - i n\right)}
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r a_1 e^{\alpha(z-l)-\delta t}\sin(\frac{r\pi z}{l})}{k_2 E l^2\left(\frac{S^2+4iE+4n}{2E}+\frac{2r^2\pi^2}{El^2}+in\right)}\\
&-\sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r b_1 e^{\alpha(z-l)-\lambda t}\sin(\frac{r\pi z}{l})}{k_2 E l^2\left(\frac{S^2+4iE+4n}{2E}+\frac{2r^2\pi^2}{El^2}-in\right)}\\
&-\frac{U}{U^*}\sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r e^{\alpha(z-l)-\delta t}\sin(\frac{r\pi z}{l})}{2r^2\pi^2+\frac{l^2}{2}\left(S^2+4iE+4n\right)}\\
&-\frac{U}{U^*}\sum_{r=1}^{\infty}\frac{(-1)^r 4\pi r e^{(\alpha z-\delta t)}\sin r\pi\left\{(1-\frac{z}{l})\right\}}{2r^2\pi^2+\frac{l^2}{2}\left(S^2+4iE+4n\right)}.
\end{aligned}
\tag{3.10}
$$

Our next job is to solve first order system. For that we have to calculate second and third order derivatives of $\overline{q}_0$. Using the resulting second and third order derivatives into Eq. (3.7) we get.

$$
\frac{d^2\overline{q}_1}{dz^2}+S\frac{d\overline{q}_1}{dz}-K\overline{q}_1=\frac{-p}{\sinh\eta l}\left[\begin{array}{c}
(\frac{a}{p-in_1}+\frac{b}{p+in_1})\left\{\begin{array}{c}(\alpha^2+\eta^2)e^{\alpha z}\sinh\eta(l-z)\\-2\alpha\eta e^{\alpha z}\cosh\eta(l-z)\end{array}\right\}\\
(\frac{c}{p-in_2}+\frac{d}{p+in_2})\left\{\begin{array}{c}(\alpha^2+\eta^2)e^{\alpha(z-l)}\sinh\eta z\\+2\alpha\eta e^{\alpha(z-l)}\cosh\eta z\end{array}\right\}\\
-\frac{U}{U^*p}\left\{\begin{array}{c}(\alpha^2+\eta^2)e^{\alpha(z-l)}\sinh\eta z\\+2\alpha\eta e^{\alpha(z-l)}\cosh\eta z\\\left(\begin{array}{c}(\alpha^2+\eta^2)e^{\alpha z}\sinh\eta(l-z)\\-2\alpha\eta e^{\alpha z}\cosh\eta(l-z)\end{array}\right)\end{array}\right\}\\
\frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\left\{\begin{array}{c}(\alpha^2+\eta^2)e^{\alpha z}\sinh\eta(l-z)\\-2\alpha\eta e^{\alpha z}\cosh\eta(l-z)\end{array}\right\}\\
\frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\left\{\begin{array}{c}(\alpha^2+\eta^2)e^{\alpha(z-l)}\sinh\eta z\\+2\alpha\eta e^{\alpha(z-l)}\cosh\eta z\end{array}\right\}
\end{array}\right]
$$

$$+\frac{\upsilon_2 S}{\upsilon_1 \sinh \eta l}\begin{bmatrix} (\frac{a}{p-in_1}+\frac{b}{p+in_1})\left\{\begin{array}{c}(\alpha^3+3\alpha\eta^2)e^{\alpha z}\sinh\eta(l-z)\\ -(3\alpha^2\eta+\eta^3)e^{\alpha z}\cosh\eta(l-z)\end{array}\right\} \\ (\frac{c}{p-in_2}+\frac{d}{p+in_2})\left\{\begin{array}{c}(\alpha^3+3\alpha\eta^2)e^{\alpha(z-l)}\sinh\eta z\\ +(\eta^3+3\alpha^2\eta)e^{\alpha(z-l)}\cosh\eta z\end{array}\right\} \\ -\frac{U}{U^*p}\left\{\begin{array}{c}(\alpha^3+3\alpha\eta^2)e^{\alpha(z-l)}\sinh\eta z\\ +(\alpha^3+3\alpha\eta^2)e^{\alpha(z-l)}\cosh\eta z\\ \left(\begin{array}{c}(\alpha^3+3\alpha\eta^2)e^{\alpha z}\sinh\eta(l-z)\\ -(\alpha^3+3\alpha\eta^2)e^{\alpha z}\cosh\eta(l-z)\end{array}\right)\end{array}\right\} \\ \frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\left\{\begin{array}{c}(\alpha^3+3\alpha\eta^2)e^{\alpha z}\sinh\eta(l-z)\\ -(3\alpha^2\eta+\eta^3)e^{\alpha z}\cosh\eta(l-z)\end{array}\right\} \\ \frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\left\{\begin{array}{c}(\alpha^3+3\alpha\eta^2)e^{\alpha(z-l)}\sinh\eta z\\ +(\eta^3+3\alpha^2\eta)e^{\alpha(z-l)}\cosh\eta z\end{array}\right\} \end{bmatrix}. \tag{3.11}$$

The above equation is non-homogeneous, second order ordinary differential equation. The general solution can be written as

$$\overline{q}_1(z,p) = \overline{q}_{1c}(z,p) + \overline{q}_{1p}(z,p), \tag{3.12}$$

where

$$\overline{q}_{1c}(z,p) = e^{\alpha z}\left[A\cosh\eta z + B\sinh\eta z\right],$$

and

$$\overline{q}_{1p}(z,p) = \frac{-p}{\sinh\eta l}\left[\begin{array}{c}
(\frac{a}{p-in_1}+\frac{b}{p+in_1})\left\{\begin{array}{c}-\frac{z(\alpha^2+\eta^2)}{2\eta}e^{\alpha z}\cosh\eta(l-z)\\ +\alpha z e^{\alpha z}\sinh\eta(l-z)\end{array}\right\}\\
+(\frac{c}{p-in_2}+\frac{d}{p+in_2})\left\{\begin{array}{c}\frac{z(\alpha^2+\eta^2)}{2\eta}e^{\alpha(z-l)}\cosh\eta z\\ +\alpha z e^{\alpha(z-l)}\sinh\eta z\end{array}\right\}\\
-\frac{U}{U^*p}\left\{\begin{array}{c}(\alpha^2+\eta^2)ze^{\alpha(z-l)}\cosh\eta z\\ +\alpha z e^{\alpha(z-l)}\sinh\eta z\\ +\left(\begin{array}{c}-\frac{z(\alpha^2+\eta^2)}{2\eta}e^{\alpha z}\cosh\eta(l-z)\\ +\alpha z e^{\alpha z}\sinh\eta(l-z)\end{array}\right)\end{array}\right\}\\
\frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\left\{\begin{array}{c}-\frac{z(\alpha^2+\eta^2)}{2\eta}e^{\alpha z}\cosh\eta(l-z)\\ +\alpha z e^{\alpha z}\sinh\eta(l-z)\end{array}\right\}\\
\frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\left\{\begin{array}{c}\frac{z(\alpha^2+\eta^2)}{2\eta}e^{\alpha(z-l)}\cosh\eta z\\ +\alpha z e^{\alpha(z-l)}\sinh\eta z\end{array}\right\}
\end{array}\right]$$

$$+\frac{\upsilon_2 S}{\upsilon_1\sinh\eta l}\left[\begin{array}{c}
(\frac{a}{p-in_1}+\frac{b}{p+in_1})\left\{\begin{array}{c}-\frac{z(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha z}\cosh\eta(l-z)\\ +\frac{z(\eta^2+3\alpha^2)}{2}e^{\alpha z}\sinh\eta(l-z)\end{array}\right\}\\
+(\frac{c}{p-in_2}+\frac{d}{p+in_2})\left\{\begin{array}{c}\frac{z(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha(z-l)}\cosh\eta z\\ +\frac{z(\eta^2+3\alpha^2)}{2}e^{\alpha(z-l)}\sinh\eta z\end{array}\right\}\\
-\frac{U}{U^*p}\left\{\begin{array}{c}\left(\begin{array}{c}\frac{z(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha(z-l)}\cosh\eta z\\ +\frac{z(\eta^2+3\alpha^2)}{2}e^{\alpha(z-l)}\sinh\eta z\end{array}\right)\\ +\left(\begin{array}{c}-\frac{z(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha z}\cosh\eta(l-z)\\ +\frac{z(\eta^2+3\alpha^2)}{2}e^{\alpha z}\sinh\eta(l-z)\end{array}\right)\end{array}\right\}\\
\frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\left\{\begin{array}{c}-\frac{z(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha z}\cosh\eta(l-z)\\ +\frac{z(\eta^2+3\alpha^2)}{2}e^{\alpha z}\sinh\eta(l-z)\end{array}\right\}\\
\frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\left\{\begin{array}{c}\frac{z(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha(z-l)}\cosh\eta z\\ +\frac{z(\eta^2+3\alpha^2)}{2}e^{\alpha(z-l)}\sinh\eta z\end{array}\right\}
\end{array}\right].$$

We can also write Eq. (3.12) as

$$
\begin{aligned}
\overline{q}_1(z,p) \;=\; & e^{\alpha z}\left[A\cosh\eta z + B\sinh\eta z\right] \\
& -\frac{p}{\sinh\eta l}\left[\begin{array}{c}
(\frac{a}{p-in_1}+\frac{b}{p+in_1})\left\{\begin{array}{c}\frac{-z(\alpha^2+\eta^2)}{2\eta}e^{\alpha z}\cosh\eta(l-z)\\ +\alpha z e^{\alpha z}\sinh\eta(l-z)\end{array}\right\} \\
+(\frac{c}{p-in_2}+\frac{d}{p+in_2})\left\{\begin{array}{c}\frac{z(\alpha^2+\eta^2)}{2\eta}e^{\alpha(z-l)}\cosh\eta z\\ +\alpha z e^{\alpha(z-l)}\sinh\eta z\end{array}\right\} \\
-\frac{U}{U^*p}\left\{\begin{array}{c}\frac{z(\alpha^2+\eta^2)}{2\eta}e^{\alpha(z-l)}\cosh\eta z\\ +\alpha z e^{\alpha(z-l)}\sinh\eta z\\ \left(\begin{array}{c}\frac{-z(\alpha^2+\eta^2)}{2\eta}e^{\alpha z}\cosh\eta(l-z)\\ +\alpha z e^{\alpha z}\sinh\eta(l-z)\end{array}\right)\end{array}\right\} \\
+\frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\left\{\begin{array}{c}-\frac{z(\alpha^2+\eta^2)}{2\eta}e^{\alpha z}\cosh\eta(l-z)\\ +\alpha z e^{\alpha z}\sinh\eta(l-z)\end{array}\right\} \\
+\frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\left\{\begin{array}{c}\frac{z(\alpha^2+\eta^2)}{2\eta}e^{\alpha(z-l)}\cosh\eta z\\ +\alpha z e^{\alpha(z-l)}\sinh\eta z\end{array}\right\}
\end{array}\right] \\
& .+\frac{\upsilon_2 S}{\upsilon_1\sinh\eta l}\left[\begin{array}{c}
(\frac{a}{p-in_1}+\frac{b}{p+in_1})\left\{\begin{array}{c}\frac{-z(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha z}\cosh\eta(l-z)\\ +\frac{z(\eta^2+3\alpha^2)}{2}e^{\alpha z}\sinh\eta(l-z)\end{array}\right\} \\
+(\frac{c}{p-in_2}+\frac{d}{p+in_2})\left\{\begin{array}{c}\frac{z(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha(z-l)}\cosh\eta z\\ +\frac{z(\eta^2+3\alpha^2)}{2}e^{\alpha(z-l)}\sinh\eta z\end{array}\right\} \\
-\frac{U}{U^*p}\left\{\begin{array}{c}\left(\begin{array}{c}\frac{z(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha(z-l)}\cosh\eta z\\ +\frac{z(\eta^2+3\alpha^2)}{2}e^{\alpha(z-l)}\sinh\eta z\end{array}\right)\\ +\left(\begin{array}{c}\frac{-z(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha z}\cosh\eta(l-z)\\ +\frac{z(\eta^2+3\alpha^2)}{2}e^{\alpha z}\sinh\eta(l-z)\end{array}\right)\end{array}\right\} \\
+\frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\left\{\begin{array}{c}\frac{-z(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha z}\cosh\eta(l-z)\\ +\frac{z(\eta^2+3\alpha^2)}{2}e^{\alpha z}\sinh\eta(l-z)\end{array}\right\} \\
+\frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\left\{\begin{array}{c}\frac{z(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha(z-l)}\cosh\eta z\\ +\frac{z(\eta^2+3\alpha^2)}{2}e^{\alpha(z-l)}\sinh\eta z\end{array}\right\}
\end{array}\right] \qquad (3.13)
\end{aligned}
$$

in which $\eta = \frac{1}{2}\left[S^2+4k\right]^{1/2}$, and $k = \left[n+iE+\frac{E}{2}p\right]$ and A and B are arbitrary constants.

With the use of conditions (3.7a)- (3.7b) we get

$$A = 0,$$

and

$$B = \frac{p}{e^{\alpha l}\sinh^2\lambda l}\begin{bmatrix} (\frac{a}{p-in_1}+\frac{b}{p+in_1})\frac{l(\alpha^2+\eta^2)}{2\eta}e^{\alpha l} \\ +(\frac{c}{p-in_2}+\frac{d}{p+in_2})\left\{\begin{array}{c}\frac{l(\alpha^2+\eta^2)}{2\eta}\cosh\eta l\\ +\alpha l\sinh\eta l\end{array}\right\} \\ -\frac{U}{U^*p}\left\{\begin{array}{c}\frac{l(\alpha^2+\eta^2)}{2\eta}\cosh\eta l+\alpha l\sinh\eta l\\ \frac{-l(\alpha^2+\eta^2)}{2\eta}e^{\alpha l}\end{array}\right\} \\ -\frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\frac{l(\alpha^2+\eta^2)}{2\eta}e^{\alpha l} \\ +\frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\left\{\begin{array}{c}\frac{l(\alpha^2+\eta^2)}{2\eta}\cosh\eta l\\ +\alpha l\sinh\eta l\end{array}\right\} \end{bmatrix}$$

$$-\frac{\upsilon_2 S}{\upsilon_1 e^{\alpha l}\sinh\eta l}\begin{bmatrix} -(\frac{a}{p-in_1}+\frac{b}{p+in_1})\left\{\frac{l(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha l}\right\} \\ +(\frac{c}{p-in_2}+\frac{d}{p+in_2})\left\{\begin{array}{c}\frac{l(\alpha^3+3\alpha\eta^2)}{2\eta}\cosh\eta l\\ +\frac{l(\eta^2+3\alpha^2)}{2}\sinh\eta l\end{array}\right\} \\ -\frac{U}{U^*p}\left\{\begin{array}{c}\left(\begin{array}{c}\frac{l(\alpha^3+3\alpha\eta^2)}{2\eta}\cosh\eta l\\ +\frac{l(\eta^2+3\alpha^2)}{2}\sinh\eta l\end{array}\right)\\ +\left(\frac{-l(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha l}\right)\end{array}\right\} \\ -\frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\left\{\frac{l(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha l}\right\} \\ +\frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\left\{\begin{array}{c}\frac{l(\alpha^3+3\alpha\eta^2)}{2\eta}\cosh\eta l\\ +\frac{l(\eta^2+3\alpha^2)}{2}\sinh\eta l\end{array}\right\} \end{bmatrix}.$$

and so Eq. (3.12) gives

$$
\begin{aligned}
\overline{q}_1(z,p) \;=\; e^{\alpha z}\sinh\eta z
&\left[\begin{array}{c}
\frac{p}{e^{\alpha l}\sinh^2\lambda l}\left[\begin{array}{c}
-(\frac{a}{p-in_1}+\frac{b}{p+in_1})\frac{-l(\alpha^2+\eta^2)}{2\eta}e^{\alpha l}\\
+(\frac{c}{p-in_2}+\frac{d}{p+in_2})\left\{\begin{array}{c}\frac{l(\alpha^2+\eta^2)}{2\eta}\cosh\eta l\\ +\alpha l\sinh\eta l\end{array}\right\}\\
-\frac{U}{U^*p}\left\{\begin{array}{c}\frac{l(\alpha^2+\eta^2)}{2\eta}\cosh\eta l+\alpha l\sinh\eta l\\ \frac{-l(\alpha^2+\eta^2)}{2\eta}e^{\alpha l}\end{array}\right\}\\
-\frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\frac{l(\alpha^2+\eta^2)}{2\eta}e^{\alpha l}\\
+\frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\left\{\begin{array}{c}\frac{l(\alpha^2+\eta^2)}{2\eta}\cosh\eta l\\ +\alpha l\sinh\eta l\end{array}\right\}
\end{array}\right]\\
-\frac{v_2S}{v_1e^{\alpha l}\sinh\eta l}\left[\begin{array}{c}
-(\frac{a}{p-in_1}+\frac{b}{p+in_1})\left\{\frac{l(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha l}\right\}\\
+(\frac{c}{p-in_2}+\frac{d}{p+in_2})\left\{\begin{array}{c}\frac{l(\alpha^3+3\alpha\eta^2)}{2\eta}\cosh\eta l\\ +\frac{l(\eta^2+3\alpha^2)}{2}\sinh\eta l\end{array}\right\}\\
-\frac{U}{U^*p}\left\{\begin{array}{c}\left(\begin{array}{c}\frac{l(\alpha^3+3\alpha\eta^2)}{2\eta}\cosh\eta l\\ +\frac{l(\eta^2+3\alpha^2)}{2}\sinh\eta l\end{array}\right)\\ +\left(\frac{-l(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha l}\right)\end{array}\right\}\\
-\frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\left\{\frac{l(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha l}\right\}\\
+\frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\left\{\begin{array}{c}\frac{l(\alpha^3+3\alpha\eta^2)}{2\eta}\cosh\eta l\\ +\frac{l(\eta^2+3\alpha^2)}{2}\sinh\eta l\end{array}\right\}
\end{array}\right]
\end{array}\right]\\
&\frac{-p}{\sinh\eta l}\left[\begin{array}{c}
(\frac{a}{p-in_1}+\frac{b}{p+in_1})\left\{\begin{array}{c}-\frac{z(\alpha^2+\eta^2)}{2\eta}e^{\alpha z}\cosh\eta(l-z)\\ +\alpha z e^{\alpha z}\sinh\eta(l-z)\end{array}\right\}\\
+(\frac{c}{p-in_2}+\frac{d}{p+in_2})\left\{\begin{array}{c}\frac{z(\alpha^2+\eta^2)}{2\eta}e^{\alpha(z-l)}\cosh\eta z\\ +\alpha z e^{\alpha(z-l)}\sinh\eta z\end{array}\right\}\\
-\frac{U}{U^*p}\left\{\begin{array}{c}(\alpha^2+\eta^2)ze^{\alpha(z-l)}\cosh\eta z\\ +\alpha z e^{\alpha(z-l)}\sinh\eta z\\ +\left(\begin{array}{c}-\frac{z(\alpha^2+\eta^2)}{2\eta}e^{\alpha z}\cosh\eta(l-z)\\ +\alpha z e^{\alpha z}\sinh\eta(l-z)\end{array}\right)\end{array}\right\}\\
\frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\left\{\begin{array}{c}-\frac{z(\alpha^2+\eta^2)}{2\eta}e^{\alpha z}\cosh\eta(l-z)\\ +\alpha z e^{\alpha z}\sinh\eta(l-z)\end{array}\right\}\\
\frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\left\{\begin{array}{c}\frac{z(\alpha^2+\eta^2)}{2\eta}e^{\alpha(z-l)}\cosh\eta z\\ +\alpha z e^{\alpha(z-l)}\sinh\eta z\end{array}\right\}
\end{array}\right]
\end{aligned}
$$

$$+\frac{\upsilon_2 S}{\upsilon_1 \sinh\eta l}\left[\begin{array}{l}
(\frac{a}{p-in_1}+\frac{b}{p+in_1})\left\{\begin{array}{c}\frac{-z(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha z}\cosh\eta(l-z)\\+\frac{z(\eta^2+3\alpha^2)}{2}e^{\alpha z}\sinh\eta(l-z)\end{array}\right\}\\
+(\frac{c}{p-in_2}+\frac{d}{p+in_2})\left\{\begin{array}{c}\frac{z(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha(z-l)}\cosh\eta z\\+\frac{z(\eta^2+3\alpha^2)}{2}e^{\alpha(z-l)}\sinh\eta z\end{array}\right\}\\
-\frac{U}{U^*p}\left\{\begin{array}{c}\left(\begin{array}{c}\frac{z(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha(z-l)}\cosh\eta z\\+\frac{z(\eta^2+3\alpha^2)}{2}e^{\alpha(z-l)}\sinh\eta z\end{array}\right)\\+\left(\begin{array}{c}\frac{-z(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha z}\cosh\eta(l-z)\\+\frac{z(\eta^2+3\alpha^2)}{2}e^{\alpha z}\sinh\eta(l-z)\end{array}\right)\end{array}\right\}\\
+\frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\left\{\begin{array}{c}\frac{-z(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha z}\cosh\eta(l-z)\\+\frac{z(\eta^2+3\alpha^2)}{2}e^{\alpha z}\sinh\eta(l-z)\end{array}\right\}\\
+\frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\left\{\begin{array}{c}\frac{z(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha(z-l)}\cosh\eta z\\+\frac{z(\eta^2+3\alpha^2)}{2}e^{\alpha(z-l)}\sinh\eta z\end{array}\right\}
\end{array}\right]. \tag{3.14}$$

Now in order to get the solution in z-plane we define inverse Laplace transform as

$$q_1(z,t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{pt}\overline{q}_1(z,t)dp.$$

Making use of Eq. (3.14) into above equation we have

$$q_1(z,t)=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}e^{\alpha z}\sinh\eta z\frac{p}{e^{\alpha l}\sinh^2\lambda l}\left[\begin{array}{c}
-(\frac{a}{p-in_1}+\frac{b}{p+in_1})\frac{l(\alpha^2+\eta^2)}{2\eta}e^{\alpha l}\\
+(\frac{c}{p-in_2}+\frac{d}{p+in_2})\left\{\begin{array}{c}\frac{l(\alpha^2+\eta^2)}{2\eta}\cosh\eta l\\+\alpha l\sinh\eta l\end{array}\right\}\\
-\frac{U}{U^*p}\left\{\begin{array}{c}\frac{l(\alpha^2+\eta^2)}{2\eta}\cosh\eta l+\alpha l\sinh\eta l\\\frac{-l(\alpha^2+\eta^2)}{2\eta}e^{\alpha l}\end{array}\right\}\\
-\frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\frac{l(\alpha^2+\eta^2)}{2\eta}e^{\alpha l}\\
+\frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\left\{\begin{array}{c}\frac{l(\alpha^2+\eta^2)}{2\eta}\cosh\eta l\\+\alpha l\sinh\eta l\end{array}\right\}
\end{array}\right]e^{pt}dp$$

$$-\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\alpha z}\sinh\eta z\frac{Sv_2}{v_1 e^{\alpha l}\sinh^2\eta l}\begin{bmatrix} -(\frac{a}{p-in_1}+\frac{b}{p+in_1})\left\{\frac{l(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha l}\right\} \\ +(\frac{c}{p-in_2}+\frac{d}{p+in_2})\left\{\begin{array}{c}\frac{l(\alpha^3+3\alpha\eta^2)}{2\eta}\cosh\eta l\\ +\frac{l(\eta^2+3\alpha^2)}{2}\sinh\eta l\end{array}\right\} \\ -\frac{U}{U^*p}\left\{\begin{array}{c}\left(\begin{array}{c}\frac{l(\alpha^3+3\alpha\eta^2)}{2\eta}\cosh\eta l\\ +\frac{l(\eta^2+3\alpha^2)}{2}\sinh\eta l\end{array}\right)\\ +\left(\frac{-l(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha l}\right)\end{array}\right\} \\ -\frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\left\{\frac{l(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha l}\right\} \\ +\frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\left\{\begin{array}{c}\frac{l(\alpha^3+3\alpha\eta^2)}{2\eta}\cosh\eta l\\ +\frac{l(\eta^2+3\alpha^2)}{2}\sinh\eta l\end{array}\right\}\end{bmatrix} e^{pt}dp$$

$$+\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{-p}{\sinh\eta l}\begin{bmatrix} (\frac{a}{p-in_1}+\frac{b}{p+in_1})\left\{\begin{array}{c}-\frac{z(\alpha^2+\eta^2)}{2\eta}e^{\alpha z}\cosh\eta(l-z)\\ +\alpha z e^{\alpha z}\sinh\eta(l-z)\end{array}\right\} \\ +(\frac{c}{p-in_2}+\frac{d}{p+in_2})\left\{\begin{array}{c}\frac{z(\alpha^2+\eta^2)}{2\eta}e^{\alpha(z-l)}\cosh\eta z\\ +\alpha z e^{\alpha(z-l)}\sinh\eta z\end{array}\right\} \\ -\frac{U}{U^*p}\left\{\begin{array}{c}(\alpha^2+\eta^2)ze^{\alpha(z-l)}\cosh\eta z\\ +\alpha z e^{\alpha(z-l)}\sinh\eta z\\ +\left(\begin{array}{c}-\frac{z(\alpha^2+\eta^2)}{2\eta}e^{\alpha z}\cosh\eta(l-z)\\ +\alpha z e^{\alpha z}\sinh\eta(l-z)\end{array}\right)\end{array}\right\} \\ \frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\left\{\begin{array}{c}-\frac{z(\alpha^2+\eta^2)}{2\eta}e^{\alpha z}\cosh\eta(l-z)\\ +\alpha z e^{\alpha z}\sinh\eta(l-z)\end{array}\right\} \\ \frac{1}{K}(\frac{a_1}{p-in}+\frac{b_1}{p+in})\left\{\begin{array}{c}\frac{z(\alpha^2+\eta^2)}{2\eta}e^{\alpha(z-l)}\cosh\eta z\\ +\alpha z e^{\alpha(z-l)}\sinh\eta z\end{array}\right\}\end{bmatrix} e^{pt}dp$$

$$+\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\upsilon_2 S}{\upsilon_1\sinh\eta l}\left[\begin{array}{l}
\left(\frac{a}{p-in_1}+\frac{b}{p+in_1}\right)\left\{\begin{array}{c}\frac{-z(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha z}\cosh\eta(l-z)\\+\frac{z(\eta^2+3\alpha^2)}{2}e^{\alpha z}\sinh\eta(l-z)\end{array}\right\}\\
+\left(\frac{c}{p-in_2}+\frac{d}{p+in_2}\right)\left\{\begin{array}{c}\frac{z(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha(z-l)}\cosh\eta z\\+\frac{z(\eta^2+3\alpha^2)}{2}e^{\alpha(z-l)}\sinh\eta z\end{array}\right\}\\
-\frac{U}{U^*p}\left\{\begin{array}{c}\left(\begin{array}{c}\frac{z(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha(z-l)}\cosh\eta z\\+\frac{z(\eta^2+3\alpha^2)}{2}e^{\alpha(z-l)}\sinh\eta z\end{array}\right)\\+\left(\begin{array}{c}\frac{-z(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha z}\cosh\eta(l-z)\\+\frac{z(\eta^2+3\alpha^2)}{2}e^{\alpha z}\sinh\eta(l-z)\end{array}\right)\end{array}\right\}\\
+\frac{1}{K}\left(\frac{a_1}{p-in}+\frac{b_1}{p+in}\right)\left\{\begin{array}{c}\frac{-z(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha z}\cosh\eta(l-z)\\+\frac{z(\eta^2+3\alpha^2)}{2}e^{\alpha z}\sinh\eta(l-z)\end{array}\right\}\\
+\frac{1}{K}\left(\frac{a_1}{p-in}+\frac{b_1}{p+in}\right)\left\{\begin{array}{c}\frac{z(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha(z-l)}\cosh\eta z\\+\frac{z(\eta^2+3\alpha^2)}{2}e^{\alpha(z-l)}\sinh\eta z\end{array}\right\}
\end{array}\right]e^{pt}dp.$$

or

$$\begin{aligned}q(z,t) &= -I_1+I_2-I_3+I_4-I_5-I_6-I_7-I_8+I_9+I_{10}+I_{11}-I_{12}\\ &\quad -I_{13}+I_{14}+I_{15}+I_{16}+I_{17}+I_{18}+I_{19}+I_{20},\end{aligned}\tag{3.15}$$

where

$$I_1=\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\left\{\left(\frac{a}{p-in_1}+\frac{b}{p+in_1}\right)\frac{lp(\alpha^2+\eta^2)}{2\eta}\frac{e^{\alpha z}\sinh\eta z}{\sinh^2\eta l}\right\}e^{pt}dp,$$

$$I_2=\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\left\{\left(\frac{c}{p-in_2}+\frac{d}{p+in_2}\right)\left(\begin{array}{c}\frac{l(\alpha^2+\eta^2)}{2\eta}\cosh\eta l\\+\alpha l\sinh\eta l\end{array}\right)\frac{pe^{\alpha z}\sinh\eta z}{e^{\alpha l}\sinh^2\eta l}\right\}e^{pt}dp,$$

$$I_3=\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\frac{U}{U^*}\left\{\left(\begin{array}{c}\frac{l(\alpha^2+\eta^2)}{2\eta}\cosh\eta l\\+\alpha l\sinh\eta l-\frac{l(\alpha^2+\eta^2)}{2\eta}e^{\alpha l}\end{array}\right)\frac{e^{\alpha z}\sinh\eta z}{e^{\alpha l}\sinh^2\eta l}\right\}e^{pt}dp,$$

$$I_4=\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\frac{S\upsilon_2}{\upsilon_1}\left\{\left(\frac{a}{p-in_1}+\frac{b}{p+in_1}\right)\frac{le^{\alpha z}\sinh\eta z}{\sinh^2\eta l}\frac{-(\alpha^3+3\alpha\eta^2)}{2\eta}\right\}e^{pt}dp,$$

$$I_5=\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\frac{S\upsilon_2}{\upsilon_1}\left\{\left(\frac{c}{p-in_2}+\frac{d}{p+in_2}\right)\frac{e^{\alpha z}\sinh\eta z}{e^{\alpha l}\sinh^2\eta l}\left(\begin{array}{c}\frac{l(\alpha^3+3\alpha\eta^2)}{2\eta}\cosh\eta l\\+\alpha\sinh\eta l\end{array}\right)\right\}e^{pt}dp,$$

$$I_6 = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \left\{ \frac{USv_2}{v_1U^*p}\frac{e^{\alpha z}\sinh\eta z}{e^{\alpha l}\sinh^2\eta l} \left(\begin{array}{c} \left(\begin{array}{c} \frac{l(\alpha^3+3\alpha\eta^2)}{2\eta}\cosh\eta l \\ +\frac{l(\eta^2+3\alpha^2)}{2}\sinh\eta l \end{array} \right) \\ -\left(\frac{l(\alpha^3+3\alpha\eta^2)}{2\eta}e^{\alpha l}\right) \end{array} \right) \right\} e^{pt}dp,$$

$$I_7 = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \left\{ \left(\frac{a}{p-in_1}+\frac{b}{p+in_1}\right)\frac{pze^{\alpha z}}{\sinh\eta l} \left(\begin{array}{c} -\frac{(\alpha^2+\eta^2)}{2\eta}\cosh\eta(l-z) \\ +\alpha\sinh\eta\,(l-z) \end{array} \right) \right\} e^{pt}dp,$$

$$I_8 = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \left\{ \left(\frac{c}{p-in_2}+\frac{d}{p+in_2}\right)\frac{pze^{\alpha(z-l)}}{\sinh\eta l} \left(\begin{array}{c} \frac{(\alpha^2+\eta^2)}{2\eta}\cosh\eta l \\ +\alpha\sinh\eta l \end{array} \right) \right\} e^{pt}dp,$$

$$I_9 = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{U}{U^*}\frac{1}{\sinh\eta l} \left\{ \begin{array}{c} ze^{\alpha(z-l)} \left(\begin{array}{c} \frac{(\alpha^2+\eta^2)}{2\eta}\cosh\eta l \\ +\alpha\sinh\eta l \end{array} \right) \\ +ze^{\alpha z} \left(\begin{array}{c} -\frac{(\alpha^2+\eta^2)}{2\eta}\cosh\eta(l-z) \\ +\alpha\sinh\eta\,(l-z) \end{array} \right) \end{array} \right\} e^{pt}dp,$$

$$I_{10} = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{Sv_2}{v_1}\left(\frac{a}{p-in_1}+\frac{b}{p+in_1}\right)\frac{ze^{\alpha z}}{\sinh\eta l} \left(\begin{array}{c} -\frac{\left(\alpha^3+3\alpha\eta^2\right)}{2\eta}\cosh\eta\,(l-z) \\ +\frac{(\eta^2+3\alpha^2)}{2}\sinh\eta(l-z) \end{array} \right) e^{pt}dp,$$

$$I_{11} = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{Sv_2}{v_1} \left\{ \left(\frac{c}{p-in_2}+\frac{d}{p+in_2}\right)\frac{ze^{\alpha(z-l)}}{\sinh\eta l} \left(\begin{array}{c} \frac{\left(\alpha^3+3\alpha\eta^2\right)}{2\eta}\cosh\eta z \\ +\frac{(\eta^2+3\alpha^2)}{2}\sinh\eta z \end{array} \right) \right\} e^{pt}dp,$$

$$I_{12} = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{SUv_2}{v_1U^*p\sinh\eta l} \left\{ \begin{array}{c} ze^{\alpha(z-l)} \left(\begin{array}{c} \frac{\left(\alpha^3+3\alpha\eta^2\right)}{2\eta}\cosh\eta z \\ +\frac{(\eta^2+3\alpha^2)}{2}\sinh\eta z \end{array} \right) \\ +ze^{\alpha z} \left(\begin{array}{c} -\frac{\left(\alpha^3+3\alpha\eta^2\right)}{2\eta}\cosh\eta\,(l-z) \\ +\frac{(\eta^2+3\alpha^2)}{2}\sinh\eta(l-z) \end{array} \right) \end{array} \right\} e^{pt}dp,$$

$$I_{13} = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \left\{ \frac{1}{k}\left(\frac{a_1}{p-in}+\frac{b_1}{p+in}\right)\frac{lp(\alpha^2+\eta^2)}{2\eta}\frac{e^{\alpha z}\sinh\eta z}{\sinh^2\eta l} \right\} e^{pt}dp,$$

$$I_{14} = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \left\{ \frac{1}{k}\left(\frac{a_1}{p-in}+\frac{b_1}{p+in}\right) \left(\begin{array}{c} \frac{l(\alpha^2+\eta^2)}{2\eta}\cosh\eta l \\ +\alpha l\sinh\eta l \end{array} \right) \frac{pe^{\alpha z}\sinh\eta z}{e^{\alpha l}\sinh^2\eta l} \right\} e^{pt}dp,$$

$$I_{15} = \frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty} \frac{Sv_2}{v_1} \left\{ \frac{1}{k}\left(\frac{a_1}{p-in}+\frac{b_1}{p+in}\right)\frac{le^{\alpha z}\sinh\eta z}{\sinh^2\lambda l}\frac{-(\alpha^3+3\alpha\eta^2)}{2\eta} \right\} e^{pt}dp,$$

$$I_{16}=\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\frac{S\upsilon_2}{\upsilon_1}\left\{\frac{1}{k}\left(\frac{a_1}{p-in}+\frac{b_1}{p+in}\right)\frac{e^{\alpha z}\sinh\eta z}{e^{\alpha l}\sinh^2\lambda l}\begin{pmatrix}\frac{l(\alpha^3+3\alpha\eta^2)}{2\eta}\cosh\eta l\\+\alpha\sinh\eta l\end{pmatrix}\right\}e^{pt}dp,$$

$$I_{17}=\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\left\{\frac{1}{k}\left(\frac{a_1}{p-in}+\frac{b_1}{p+in}\right)\frac{pze^{\alpha z}}{\sinh\eta l}\begin{pmatrix}-\frac{(\alpha^2+\eta^2)}{2\eta}\cosh\eta(l-z)\\+\alpha\sinh\eta\,(l-z)\end{pmatrix}\right\}e^{pt}dp,$$

$$I_{18}=\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\left\{\frac{1}{k}\left(\frac{a_1}{p-in}+\frac{b_1}{p+in}\right)\frac{pze^{\alpha(z-l)}}{\sinh\eta l}\begin{pmatrix}\frac{(\alpha^2+\eta^2)}{2\eta}\cosh\eta l\\+\alpha\sinh\eta l\end{pmatrix}\right\}e^{pt}dp,$$

$$I_{19}=\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\frac{S\upsilon_2}{\upsilon_1}\left\{\frac{1}{k}\left(\frac{a_1}{p-in}+\frac{b_1}{p+in}\right)\frac{ze^{\alpha z}}{\sinh\eta l}\begin{pmatrix}-\frac{\left(\alpha^3+3\alpha\eta^2\right)}{2\eta}\cosh\eta\,(l-z)\\+\frac{(\eta^2+3\alpha^2)}{2}\sinh\eta(l-z)\end{pmatrix}\right\}e^{pt}dp,$$

$$I_{20}=\frac{1}{2\pi i}\int_{r-i\infty}^{r+i\infty}\frac{S\upsilon_2}{\upsilon_1}\left\{\frac{1}{k}\left(\frac{a_1}{p-in}+\frac{b_1}{p+in}\right)\frac{ze^{\alpha(z-l)}}{\sinh\eta l}\begin{pmatrix}\frac{\left(\alpha^3+3\alpha\eta^2\right)}{2\eta}\cosh\eta z\\+\frac{(\eta^2+3\alpha^2)}{2}\sinh\eta z\end{pmatrix}\right\}e^{pt}dp,$$

For the solution of the above integrals, we use complex variable theory. These integrals are evaluated using a similar procedure as in chapter-2. Thus using residue theorem we can get from Eq. (3.14) as

$$\begin{aligned}q_1(z,t) &= -al\left[A+\sum_{r=1}^{\infty}e^{\alpha z-\delta t}\left(\widetilde{A}_1-\widetilde{A}\right)\right]-bl\left[B+\sum_{r=1}^{\infty}e^{\alpha z-\delta t}\left(\widetilde{B}+\widetilde{B}_1\right)\right]\\&\quad+\frac{lc}{e^{\alpha l}}\left[(C+C^*)+\sum_{r=1}^{\infty}(-1)^r e^{\alpha z-\delta t}\left(\widetilde{C}+\widetilde{C}_1+\widetilde{C}_2\right)\right]\\&\quad-\frac{ld}{e^{\alpha l}}\left[(D+D_1)-\sum_{r=1}^{\infty}(-1)^r e^{\alpha z-\delta t}\left(\widetilde{D}+\widetilde{D}_1+\widetilde{D}_2\right)\right]\\&\quad-\frac{Ul}{U^*e^{\alpha l}}\left[\sum_{r=1}^{\infty}(-1)^r e^{\alpha z-\delta t}\left(\widetilde{L}+\widetilde{L}_1+\widetilde{L}_2+\widetilde{L}_3+\widetilde{L}_4\right)\right]\end{aligned}$$

$$+\frac{alS\alpha v_2}{v_1}\left[A_1+\sum_{r=1}^{\infty}(-1)^r\,e^{\alpha z-\delta t}\left(\widetilde{A}_2+\widetilde{A}_3\right)\right]$$
$$+\frac{bSl\alpha v_2}{v_1}\left[B_1+\sum_{r=1}^{\infty}(-1)^r\,e^{\alpha z-\delta t}\left(\widetilde{B}_2+\widetilde{B}_3\right)\right]$$
$$-\frac{Slcv_2}{v_1e^{\alpha l}}\left[C_1+C_1^*+\sum_{r=1}^{\infty}(-1)^r\,e^{\alpha z-\delta t}\left(\widetilde{C}_3+\widetilde{C}_4+\widetilde{C}_5\right)\right]$$
$$-\frac{Sldv_2}{v_1e^{\alpha l}}\left[D_1+D_1^*+\sum_{r=1}^{\infty}(-1)^r\,e^{\alpha z-\delta t}\left(\widetilde{D}_3+\widetilde{D}_4+\widetilde{D}_5\right)\right]$$

$$-\frac{lSUv_2e^{\alpha z}}{v_1U^*e^{\alpha l}}\left[L+L_1+\sum_{r=1}^{\infty}(-1)^r\,e^{-\delta t}\left(\widetilde{L}_5+\widetilde{L}_6+\widetilde{L}_7\right)\right]$$
$$-\frac{USv_2e^{\alpha z}}{v_1U^*}\left[L_2+\sum_{r=1}^{\infty}(-1)^r\,e^{-\delta t}\left(\widetilde{L}_8+\widetilde{L}_9\right)\right]$$

$$zae^{\alpha z}\left[A_2+\sum_{r=1}^{\infty}(-1)^r\,e^{-\delta t}\widetilde{A}_4\right]-zbe^{\alpha z}\left[B_2+\sum_{r=1}^{\infty}(-1)^r\,e^{-\delta t}\widetilde{B}_4\right]$$
$$-zae^{\alpha(z-l)}\left[A_3+\sum_{r=1}^{\infty}(-1)^r\,e^{-\delta t}\widetilde{A}_5\right]-zbe^{\alpha(z-l)}\left[B_3+\sum_{r=1}^{\infty}(-1)^r\,e^{-\delta t}\widetilde{B}_5\right]$$
$$+\frac{zUe^{\alpha(z-l)}}{U^*}\left[\sum_{r=1}^{\infty}(-1)^r\,e^{-\delta t}\widetilde{L}_{10}\right]+\frac{zUe^{\alpha z}}{U^*}\left[\sum_{r=1}^{\infty}(-1)^r\,e^{-\delta t}\widetilde{L}_{11}\right]$$

where

$$
\begin{aligned}
A &= \frac{in_1\left(\alpha^2+2\mu_1^2\right)}{2\mu_1}\frac{e^{\alpha z+in_1 t}\sinh\mu_1 z}{\sinh^2\mu_1 l},\\
A_1 &= \frac{\left(\alpha^2+3\mu_1^2\right)}{\mu_1}\frac{e^{\alpha z+in_1 t}\sinh\mu_1 z}{\sinh^2\mu_1 l},\\
A_2 &= \frac{in_1 e^{in_1 t}}{\sinh\mu_1 l}\left\{\frac{\alpha^2+\mu_1^2}{2\mu_1}\cosh\mu_1\left(l-z\right)+\alpha\sinh\mu_1\left(l-z\right)\right\},\\
A_3 &= \frac{in_2 e^{in_2 t}}{\sinh\mu_2 l}\left\{\frac{\alpha^2+\mu_2^2}{2\mu_2}\cosh\mu_2 l+\alpha\sinh\mu_2 l\right\},\\
A_4 &= \frac{e^{in_1 t}}{\sinh\mu_1 l}\left\{\frac{-\left(\alpha 3+3\alpha\mu_1^2\right)}{2\mu_1}\cosh\mu_1\left(l-z\right)+\frac{\alpha^2+\mu_1^2}{2}\sinh\mu_1\left(l-z\right)\right\},\\
A_5 &= \frac{e^{in_2 t}}{\sinh\mu_1 l}\left\{\frac{\alpha^3+3\alpha\mu_1^2}{2\mu_1}\cosh\mu_1 z+\frac{\alpha^2+\mu_1^2}{2}\sinh\mu_1 z\right\},
\end{aligned}
$$

$$
\begin{aligned}
B &= \frac{in_1\left(\alpha^2-2\mu_1'^2\right)}{2\mu_1'}\frac{e^{\alpha z+in_1 t}\sinh\mu_1' z}{\sinh^2\mu_1' l},\\
B_1 &= \frac{\left(\alpha^2+3\mu_1'^2\right)}{\mu_1'}\frac{e^{\alpha z+in_1 t}\sinh\mu_1' z}{\sinh^2\mu_1' l},\\
B_2 &= \frac{in_1 e^{-in_1 t}}{\sinh\mu_1' l}\left\{\frac{\alpha^2+\mu_1'^2}{2\mu_1'}\cosh\mu_1'\left(l-z\right)+\alpha\sinh\mu_1'\left(l-z\right)\right\},\\
B_3 &= \frac{-in_2 e^{-in_2 t}}{\sinh\mu_2' l}\left\{\frac{\alpha^2+\mu_2'^2}{2\mu_2}\cosh\mu_2' l+\alpha\sinh\mu_2' l\right\},\\
B_4 &= \frac{e^{-in_1 t}}{\sinh\mu_1' l}\left\{\frac{-\left(\alpha^3+3\alpha\mu_1'^2\right)}{2\mu_1'}\cosh\mu_1'\left(l-z\right)+\frac{\alpha^2+\mu_1'^2}{2}\sinh\mu_1'\left(l-z\right)\right\},\\
B_5 &= \frac{e^{-in_2 t}}{\sinh\mu_1' l}\left\{\frac{\alpha^3+3\alpha\mu_1'^2}{2\mu_1'}\cosh\mu_1' z+\frac{\alpha^2+\mu_1'^2}{2}\sinh\mu_1' z\right\},
\end{aligned}
$$

$$
\begin{aligned}
C &= \frac{in_2\left(\alpha^2+2\mu_2^2\right)}{2\mu_2}\frac{e^{\alpha z+in_2 t}\sinh\mu_2 z}{\sinh^2\mu_2 l},\\
C_1 &= \frac{\left(\alpha^3+3\alpha\mu_2^2\right)}{2\mu_2}\frac{e^{\alpha z+in_2 t}\sinh\mu_2 z\cosh\mu_2 l}{\sinh^2\mu_2 l},\\
C^* &= \frac{in_2\alpha e^{\alpha z+in_2 t}\sinh\mu_2 z}{\sinh^2\mu_2 l},
\end{aligned}
$$

$$
\begin{aligned}
D &= \frac{in_2\left(\alpha^2-2\mu_2'^2\right)}{2\mu_2'}\frac{e^{\alpha z-n_2t}\sinh\mu_2'z}{\sinh^2\mu_2'l},\\
D_1 &= \frac{\left(\alpha^3+3\alpha\mu_2'^2\right)}{2\mu_2'}\frac{e^{\alpha z-n_2t}\sinh\mu_2'z\cosh\mu_2'l}{\sinh^2\mu_2'l},\\
D_1^* &= \frac{\alpha e^{in_2t}\sinh\mu_2'z}{\sinh^2\mu_2'l},
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{A} &= \frac{\alpha^2}{2}\left(\frac{(\delta+in_1)\left(\left(\frac{l}{\pi r}+\frac{Epl^3}{4\pi^3r^3}-\frac{\delta tl}{\pi r}\right)\sin\frac{r\pi z}{l}+\frac{z\delta El^2}{2r^2\pi^2}\cos\frac{r\pi z}{l}\right)+\frac{\delta l}{r\pi}\sin\frac{r\pi z}{l}}{(\delta+in_1)^2}\right),\\
\widetilde{A}_1 &= \frac{1}{2}\left(\frac{-(\delta+in_1)\left(\left(-\frac{\pi r}{l}-\frac{E\delta l}{2\pi r}+\frac{\delta t\pi r}{l}\right)\sin\frac{r\pi z}{l}-\frac{z\delta E}{2}\cos\frac{r\pi z}{l}\right)+\frac{\delta\pi r}{l}\sin\frac{r\pi z}{l}}{(\delta+in_1)^2}\right),\\
\widetilde{A}_2 &= \alpha\left(\frac{-(\delta+in_1)\left(\left(\frac{El^3}{2\pi^3r^3}+\frac{tl}{\pi r}\right)\sin\frac{r\pi z}{l}-\frac{zEl^2}{2r^2\pi^2}\cos\frac{r\pi z}{l}\right)-\frac{l}{r\pi}\sin\frac{r\pi z}{l}}{(\delta+in_1)^2}\right),\\
\widetilde{A}_3 &= 3\left(\frac{-(\delta+in_1)\left(\left(\frac{El}{2\pi r}-\frac{t\pi r}{l}\right)\sin\frac{r\pi z}{l}-\frac{zE}{2}\cos\frac{r\pi z}{l}\right)+\frac{\pi r}{l}\sin\frac{r\pi z}{l}}{(\delta+in_1)^2}\right),\\
\widetilde{A}_4 &= \frac{4\pi r\delta}{El^2(\delta+in_1)}\left\{\frac{\left(r^2\pi^2-\alpha^2l^2\right)}{2\pi rl}\cos r\pi\left(1-\frac{z}{l}\right)+\alpha\sin r\pi\left(1-\frac{z}{l}\right)\right\},\\
\widetilde{A}_5 &= \frac{2\delta}{El^3(\delta+in_2)}\left(r^2\pi^2-\alpha^2l^2\right),\\
\widetilde{A}_6 &= \frac{4\pi r}{El^2(\delta+in_1)}\left\{\frac{\left(\alpha^2l^2-3\alpha r^2\pi^2\right)}{2\pi rl}\cos r\pi\left(1-\frac{z}{l}\right)+\frac{\left(\alpha^2l^2-r^2\pi^2\right)}{2l^2}\sin r\pi\left(1-\frac{z}{l}\right)\right\},\\
\widetilde{A}_7 &= \frac{4\pi r}{El^2(\delta+in_2)}\left\{\frac{\left(\alpha^2l^2-3\alpha r^2\pi^2\right)}{2\pi rl}\cos\frac{r\pi z}{l}+\frac{\left(r^2\pi^2-\alpha^2l^2\right)}{2l^2}\sin\frac{r\pi z}{l}\right\},
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{B} &= \frac{\alpha^2}{2}\left(\frac{(-\delta+in_1)\left(\left(\frac{El^3}{2\pi^3r^3}+\frac{tl}{\pi r}\right)\sin\frac{r\pi z}{l}-\frac{zEl^2}{2r^2\pi^2}\cos\frac{r\pi z}{l}\right)-\frac{l}{r\pi}\sin\frac{r\pi z}{l}}{(\delta+in_1)^2}\right),\\
\widetilde{B}_1 &= \frac{1}{2}\left(\frac{(-\delta+in_1)\left(\left(-\frac{\pi r}{l}-\frac{E\delta l}{2\pi r}+\frac{\delta t\pi r}{l}\right)\sin\frac{r\pi z}{l}-\frac{z\delta E}{2}\cos\frac{r\pi z}{l}\right)+\frac{\delta\pi r}{l}\sin\frac{r\pi z}{l}}{(-\delta+in_1)^2}\right),\\
\widetilde{B}_2 &= \alpha^2\left(\frac{(-\delta+in_1)\left(\left(\frac{El^3}{2\pi^3r^3}+\frac{tl}{\pi r}\right)\sin\frac{r\pi z}{l}-\frac{zEl^2}{2r^2\pi^2}\cos\frac{r\pi z}{l}\right)-\frac{l}{r\pi}\sin\frac{r\pi z}{l}}{(\delta+in_1)^2}\right),\\
\widetilde{B}_3 &= 3\left(\frac{(-\delta+in_1)\left(\left(\frac{El}{2\pi r}-\frac{t\pi r}{l}\right)\sin\frac{r\pi z}{l}-\frac{zE}{2}\cos\frac{r\pi z}{l}\right)+\frac{\pi r}{l}\sin\frac{r\pi z}{l}}{(-\delta+in_1)^2}\right),\\
\widetilde{B}_4 &= \frac{4\pi r\delta}{El^2(-\delta+in_1)}\left\{\frac{\left(r^2\pi^2-\alpha^2l^2\right)}{2\pi rl}\cos r\pi\left(1-\frac{z}{l}\right)+\alpha\sin r\pi\left(1-\frac{z}{l}\right)\right\},\\
\widetilde{B}_5 &= \frac{2\delta}{El^3(\delta-in_2)}\left(r^2\pi^2-\alpha^2l^2\right),\\
\widetilde{B}_6 &= \frac{4\pi r}{El^2(-\delta+in_1)}\left\{\frac{\left(\alpha^3l^2-3\alpha r^2\pi^2\right)}{2\pi rl}\cos r\pi\left(1-\frac{z}{l}\right)+\frac{\left(\alpha^2l^2-r^2\pi^2\right)}{2l^2}\sin r\pi\left(1-\frac{z}{l}\right)\right\},\\
\widetilde{B}_7 &= \frac{4\pi r}{El^2(-\delta+in_2)}\left\{\frac{\left(3\alpha r^2\pi^2-\alpha^2l^2\right)}{2\pi rl}\cos\frac{r\pi z}{l}+\frac{\left(\alpha^2l^2-r^2\pi^2\right)}{2l^2}\sin\frac{r\pi z}{l}\right\},
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{C} &= \frac{\alpha^2}{2}\left(\frac{-(\delta+in_2)\left(\left(\frac{l}{\pi r}+\frac{Epl^3}{4\pi^3r^3}-\frac{\delta tl}{\pi r}\right)\sin\frac{r\pi z}{l}+\frac{z\delta El^2}{2r^2\pi^2}\cos\frac{r\pi z}{l}\right)+\frac{\delta l}{r\pi}\sin\frac{r\pi z}{l}}{(\delta+in_2)^2}\right),\\
\widetilde{C}_1 &= \frac{1}{2}\left(\frac{-(\delta+in_2)\left(\left(-\frac{\pi r}{l}-\frac{E\delta l}{2\pi r}+\frac{\delta t\pi r}{l}\right)\sin\frac{r\pi z}{l}-\frac{z\delta E}{2}\cos\frac{r\pi z}{l}\right)-\frac{\delta\pi r}{l}\sin\frac{r\pi z}{l}}{(\delta+in_2)^2}\right),\\
\widetilde{C}_2 &= \frac{4\alpha\pi r\delta\sin\frac{r\pi z}{l}}{El^2(\delta+in_2)},\\
\widetilde{C}_3 &= \frac{\alpha^3}{2}\left(\frac{-(\delta+in_2)\left(\left(\frac{El^3}{2\pi^3r^3}+\frac{tl}{\pi r}\right)\sin\frac{r\pi z}{l}-\frac{zEl^2}{2r^2\pi^2}\cos\frac{r\pi z}{l}\right)-\frac{l}{r\pi}\sin\frac{r\pi z}{l}}{(\delta+in_2)^2}\right),\\
\widetilde{C}_4 &= \frac{3\alpha}{2}\left(\frac{-(\delta+in_2)\left(\left(\frac{El}{2\pi r}-\frac{t\pi r}{l}\right)\sin\frac{r\pi z}{l}-\frac{zE}{2}\cos\frac{r\pi z}{l}\right)+\frac{\pi r}{l}\sin\frac{r\pi z}{l}}{(\delta+in_2)^2}\right),\\
\widetilde{C}_5 &= \frac{4\alpha\pi r\sin\frac{r\pi z}{l}}{El^2(-\delta-in_2)},
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{D} &= \frac{\alpha^2}{2}\left(\frac{(-\delta+in_2)\left(\left(\frac{l}{\pi r}+\frac{Epl^3}{4\pi^3r^3}-\frac{\delta tl}{\pi r}\right)\sin\frac{r\pi z}{l}+\frac{z\delta El^2}{2r^2\pi^2}\cos\frac{r\pi z}{l}\right)+\frac{\delta l}{r\pi}\sin\frac{r\pi z}{l}}{(-\delta+in_2)^2}\right),\\
\widetilde{D}_1 &= \frac{1}{2}\left(\frac{(-\delta+in_2)\left(\left(-\frac{\pi r}{l}-\frac{E\delta l}{2\pi r}+\frac{\delta t\pi r}{l}\right)\sin\frac{r\pi z}{l}-\frac{z\delta E}{2}\cos\frac{r\pi z}{l}\right)-\frac{\delta\pi r}{l}\sin\frac{r\pi z}{l}}{(-\delta+in_2)^2}\right),\\
\widetilde{D}_2 &= \frac{4\alpha\pi r\delta\sin\frac{r\pi z}{l}}{El^2(-\delta+in_2)},\\
\widetilde{D}_3 &= \frac{\alpha^3}{2}\left(\frac{(-\delta+in_2)\left(\left(\frac{El^3}{2\pi^3r^3}+\frac{tl}{\pi r}\right)\sin\frac{r\pi z}{l}-\frac{zEl^2}{2r^2\pi^2}\cos\frac{r\pi z}{l}\right)-\frac{l}{r\pi}\sin\frac{r\pi z}{l}}{(\delta+in_2)^2}\right),\\
\widetilde{D}_4 &= \frac{3\alpha}{2}\left(\frac{(-\delta+in_2)\left(\left(\frac{El}{2\pi r}-\frac{t\pi r}{l}\right)\sin\frac{r\pi z}{l}-\frac{zE}{2}\cos\frac{r\pi z}{l}\right)+\frac{\pi r}{l}\sin\frac{r\pi z}{l}}{(-\delta+in_2)^2}\right),\\
\widetilde{D}_5 &= \frac{4\alpha\pi r\sin\frac{r\pi z}{l}}{El^2(-\delta+in_2)},
\end{aligned}
$$

$$
\begin{aligned}
L &= \left(\frac{\alpha^3+3\alpha\mu_3^2}{2\mu_3}\right)\frac{\sinh\mu_3 z\cosh\mu_3 l}{\sinh^2\mu_3 l},\\
L_1 &= \left(\frac{\alpha^3+\mu_3^2}{2\mu_3}\right)\frac{\sinh\mu_3 z}{\sinh\mu_3 l},\\
L_2 &= \left(\frac{\alpha^3+3\alpha\mu_3^2}{2\mu_3}\right)\frac{\sinh\mu_3 z}{\sinh^2\mu_3 l},\\
L_3 &= \frac{1}{\sinh\mu_3 l}\left\{\frac{\alpha^3+3\alpha\mu_3^2}{2\mu_3}\cosh\mu_3 z+\frac{\alpha^2+\mu_3^2}{2}\sinh\mu_3 z\right\},\\
L_4 &= \frac{1}{\sinh\mu_3 l}\left\{\frac{-\left(\alpha 3+3\alpha\mu_3^2\right)}{2\mu_3}\cosh\mu_3(l-z)+\frac{\alpha^2+\mu_3^2}{2}\sinh\mu_3(l-z)\right\},
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{L} &= \frac{\alpha^2}{2}\left\{\left(\frac{El^3}{2\pi^3 r^3}+\frac{tl}{r\pi}\right)\sin\frac{r\pi z}{l}-\frac{zEl^2}{2r^2\pi^2}\cos\frac{r\pi z}{l}\right\},\\
\widetilde{L}_1 &= \frac{1}{2}\left\{\left(\frac{El}{2\pi r}+\frac{t\pi r}{l}\right)\sin\frac{r\pi z}{l}+\frac{zE}{2}\cos\frac{r\pi z}{l}\right\},\\
\widetilde{L}_2 &= \frac{4\pi r\sin\frac{r\pi z}{l}}{El^2},\\
\widetilde{L}_3 &= \frac{\alpha^2 e^{\alpha l}}{2}\left\{\left(\frac{El^3}{2\pi^3 r^3}+\frac{tl}{r\pi}\right)\sin\frac{r\pi z}{l}-\frac{zEl^2}{2r^2\pi^2}\cos\frac{r\pi z}{l}\right\},\\
\widetilde{L}_4 &= \frac{e^{\alpha l}}{2}\left\{\left(\frac{El}{2\pi r}+\frac{t\pi r}{l}\right)\sin\frac{r\pi z}{l}+\frac{zE}{2}\cos\frac{r\pi z}{l}\right\},\\
\widetilde{L}_5 &= \frac{\alpha^3}{2}\left(\frac{-\delta\left(\left(\frac{El^3}{2\pi^3 r^3}+\frac{tl}{\pi r}\right)\sin\frac{r\pi z}{l}-\frac{zEl^2}{2r^2\pi^2}\cos\frac{r\pi z}{l}\right)-\frac{l}{r\pi}\sin\frac{r\pi z}{l}}{\delta^2}\right),\\
\widetilde{L}_6 &= \frac{3\alpha}{2}\left(\frac{-\delta\left(\left(\frac{El}{2\pi r}-\frac{t\pi r}{l}\right)\sin\frac{r\pi z}{l}-\frac{zE}{2}\cos\frac{r\pi z}{l}\right)+\frac{\pi r}{l}\sin\frac{r\pi z}{l}}{\delta^2}\right),\\
\widetilde{L}_7 &= \frac{2\left(\alpha^2 l^2-r^2\pi^2\right)}{i\delta El^3},\\
\widetilde{L}_8 &= \frac{\alpha^3}{2}\left(\frac{-\delta\left(\left(\frac{El^3}{2\pi^3 r^3}+\frac{tl}{\pi r}\right)\sin\frac{r\pi z}{l}-\frac{zEl^2}{2r^2\pi^2}\cos\frac{r\pi z}{l}\right)-\frac{l}{r\pi}\sin\frac{r\pi z}{l}}{\delta^2}\right),\\
\widetilde{L}_9 &= \frac{3\alpha}{2}\left(\frac{-\delta\left(\left(\frac{El}{2\pi r}-\frac{t\pi r}{l}\right)\sin\frac{r\pi z}{l}-\frac{zE}{2}\cos\frac{r\pi z}{l}\right)+\frac{\pi r}{l}\sin\frac{r\pi z}{l}}{\delta^2}\right),\\
\widetilde{L}_{10} &= \frac{4\pi r}{El^2}\left\{\frac{\left(r^2\pi^2-\alpha^2 l^2\right)}{2\pi rl}\cos\frac{r\pi z}{l}+\alpha\sin\frac{r\pi z}{l}\right\},\\
\widetilde{L}_{11} &= \frac{4\pi r}{El^2}\left\{\frac{\left(\alpha^2 l^2-r^2\pi^2\right)}{2\pi rl}\cos r\pi\left(1-\frac{z}{l}\right)+\alpha\sin r\pi\left(1-\frac{z}{l}\right)\right\},\\
\widetilde{L}_{12} &= \frac{4\pi r}{\delta El^2}\left\{\frac{\left(3\alpha r^2\pi^2-\alpha^2 l^2\right)}{2\pi rl}\cos\frac{r\pi z}{l}+\frac{\left(\alpha^2 l^2-r^2\pi^2\right)}{2l^2}\sin\frac{r\pi z}{l}\right\},\\
\widetilde{L}_{13} &= \frac{4\pi r}{\delta El^2}\left\{\frac{\left(\alpha^3 l^2-3\alpha r^2\pi^2\right)}{2\pi rl}\cos r\pi\left(1-\frac{z}{l}\right)+\frac{\left(\alpha^2 l^2-r^2\pi^2\right)}{2l^2}\sin r\pi\left(1-\frac{z}{l}\right)\right\},
\end{aligned}
$$

$$
\begin{aligned}
\mu_r &= \frac{1}{2}\left[S^2+4n+2iE(2+n_r)\right]^{\frac{1}{2}},\ r=1,2,\\
\mu_r' &= \frac{1}{2}\left[S^2+4n+2iE(2-n_r)\right]^{\frac{1}{2}},\ r=1,2,\\
\mu_3 &= \frac{1}{2}\left[S^2+4iE+4n\right]^{\frac{1}{2}},\\
\delta &= \left(\frac{S^2+4iE+4n}{2E}+\frac{2r^2\pi^2}{El^2}\right),\\
\delta' &= \left(\frac{S^2+4iE+4n}{2E}\right).
\end{aligned}
$$

$$
\begin{aligned}
\mu_r &= \frac{1}{2}\left[s^2+4n+2iE(2+n_r)\right]^{\frac{1}{2}},\ r=1,2.\\
\mu_r' &= \frac{1}{2}\left[s^2+4n+2iE(2-n_r)\right]^{\frac{1}{2}},\ r=1,2.\\
\mu_3 &= \frac{1}{2}\left[s^2+4iE+4n\right]^{\frac{1}{2}},\\
\mu_4 &= \frac{1}{2}\left[s^2+4n+2iE(2+n)\right]^{\frac{1}{2}},\ k=k_1,\\
k_1 &= \left[n+iE+\frac{inE}{2}\right],\\
k_1' &= \left[n+iE-\frac{inE}{2}\right],\\
p &= -\delta=-\left(\frac{s^2+4iE+4n}{2E}+\frac{2r^2\pi^2}{El^2}\right),\ k=k_2,\\
k_2 &= \left[n+iE-\frac{E}{2}\delta\right],
\end{aligned}
$$

and blowing solution can be obtained by replacing S with $-S$.

3.4 Summary and Conclusion

The flow due to oscillating porous plates and imposed pressure gradient is studied in the presence of a transverse magnetic field. The analytic solution for hydromagnetic flow of a second grade fluid in a rotating frame of reference is given by taking material parameter as a perturbation parameter. The exact solutions for the steady and transient states are obtained. It is shown that the transient effects decay much in the presence of magnetic field and the steady state is achieved earlier when compared to hydrodynamic case. The inclusion of the non-Newtonian characteristics along with considering the case of suction (or injection) reveal

interesting phenomena. It is found in second grade fluid that steady asymptotic solutions are possible for all values of frequencies including the resonant frequency for suction (or injection).

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