A "deregularized" proof of the El-Zahar conjecture for large graphs

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Abstract

A classical conjecture of El-Zahar states that if H is a graph consisting of r vertex disjoint cycles of length $n_1, n_2, \ldots, n_r$ satisfying $n_1 + n_2 + \ldots + n_r = n$, and G is a graph on n vertices with minimum degree at least $\sum_{i=1}^{r} \lceil n_i/2 \rceil$, then G contains H as a subgraph. A proof of this conjecture for graphs with $n \geq n_0$ was announced by Abbasi using the Regularity Lemma-Blow-up Lemma method. In this paper we give a new, "deregularized" proof of the conjecture for large graphs that avoids the use of the Regularity Lemma, and thus the resulting n_0 is much smaller.

1 Introduction

The vertex-set and the edge-set of the graph G is denoted by $V(G)$ and $E(G)$. K_n is the complete graph on n vertices, $K_{r+1}(t)$ is the complete $(r+1)$-partite graph where each class contains t vertices and $K_2(t) = K(t,t)$ is the complete bipartite graph between two vertex classes of size t. C_l (P_l) denotes the cycle (path) on l vertices. We denote by (A, B, E) a bipartite graph $G = (V, E)$, where $V = A + B$, and $E \subset A \times B$. For a graph G and a subset U of its vertices, $G|_U$ is the restriction of G to U. The set of neighbors of $v \in V$ is $N(v)$. Hence the size of $N(v)$ is $|N(v)| = deg(v) = deg_G(v)$, the degree of v. The minimum degree is denoted by $\delta(G)$ and the maximum degree by $\Delta(G)$ in a graph G. When A, B are subsets of $V(G)$, we denote by $e(A, B)$ the number of edges of G with one endpoint in A and the other in B and $d(A, B) = e(A, B)/|A||B|$. In particular, we write $deg(v, U) = e(\{v\}, U)$ for the number of edges from v to U. For a graph $G = (V, E)$ on n vertices $d(G) = |E|/\binom{n}{2}$ and G is γ-*dense* if $d(G) \geq \gamma$. A bipartite graph $G = (A, B)$ is γ-*dense* if $d(A, B) \geq \gamma$. If a graph is not γ-dense, then it is γ-*sparse*. Throughout the paper log denotes the base 2 logarithm.

A classical conjecture of El-Zahar states the following.

Conjecture 1 (El-Zahar conjecture). *Let H be a graph consisting of r vertex disjoint cycles of length $n_1, n_2, \ldots, n_r$ satisfying $n_1 + n_2 + \ldots + n_r = n$, and G be a graph on n vertices with minimum degree at least $\sum_{i=1}^{r} \lceil n_i/2 \rceil$, then G contains H as a subgraph.*

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Note that the graph $K_{k-1} + K(\lceil \frac{n-k+1}{2} \rceil, \lceil \frac{n-k+1}{2} \rceil)$ (where the + operation indicates a complete bipartite graph between the two graphs) has minimum degree $(n+k-1)/2$ but contains no k vertex disjoint odd length cycles. Thus, the conjecture is best possible. This beautiful conjecture has generated a lot of attention. El-Zahar proved the conjecture for $r = 2$ in [11]. The case that each $n_i = 3$ (i.e. we only have triangles) follows from a result of Corrádi and Hajnal [9]. Wang [27] verified the conjecture for arbitrary n_1 and $n_i = 3, i \geq 2$. The case that each $n_i = 4$ (i.e. we only have 4-cycles) is an old conjecture of Erdős and Faudree [13] (see [4], [17], [20], [24] and [28] for results related to this special case). For the case of triangles and quadrilaterals see [29]. In general it was proved in [2] and in [3] that $\delta(G) \geq 2n/3$ implies the desired conclusion; note that this is a special case of the Bollobás-Eldridge conjecture (see [6]). In [16] Johansson has shown that an El-Zahar type condition implies path factors.

Finally Abbasi announced a proof of Conjecture 1 for graphs with $n \geq n_0$ in [1]. The proof used the Regularity Lemma-Blow-up Lemma method ([26], [19]) and thus the resulting n_0 was quite large (a tower function).

The main purpose of this paper is to give a new, "deregularized" proof of the El-Zahar conjecture for large graphs that avoids the use of the Regularity Lemma and thus we obtain a much smaller n_0 (although we do not compute the actual n_0, it is exponential instead of a tower function). We prove the theorem in the following more convenient form.

Theorem 1. *There exists an n_0 such that the following holds. Let H be a graph consisting of r vertex disjoint cycles of length $n_1, n_2, \ldots, n_r$ satisfying $n_1 + n_2 + \ldots + n_r = n \geq n_0$, where the number of odd cycles is denoted by k. If G is a graph on n vertices with*

$$\delta(G) \geq \frac{n+k}{2}, \tag{1}$$

then G contains H as a subgraph.

2 Tools

2.1 Complete bipartite and tripartite subgraphs

In [1] Abbasi used the Regularity Lemma [26], however, here we use a more elementary approach using only the Kővári-Sós-Turán bound [21]. This is part of a new direction to "deregularize" this type of proofs, namely to replace the Regularity Lemma with more elementary classical extremal graph theoretic results such as the Kővári-Sós-Turán bound while maintaining some other elements of the proof (see e.g. [15], [22]).

An easy consequence of the Kovari-Sos-Turan theorem [21] is the following lemma.

Lemma 2. *For every $\eta > 0$ there is a constant $c_1 > 0$ such that if G is a graph on n vertices with $|E(G)| \geq \eta n^2$, then G contains a $K_2(c_1\sqrt{\log n})$.*

The following lemma follows by a standard argument from a result of Erdős [12].

Lemma 3. *For every $\eta > 0$ there is a constant $c_2 > 0$ such that if G is a graph on n vertices containing ηn^3 triangles, then G contains a $K_3(c_2\sqrt{\log n})$.*

Let us remark that Bollobás, Erdős and Simonovits [7] proved a much stronger result of this kind, but we will not use it, as Lemma 3 is good enough for our purposes.

In the proof we are going to deal with 3 different types of graphs, the Extremal Case 1, Extremal Case 2 and α-non-extremal graphs, (see the definitions below):

Extremal Case 1 (EC1) with parameter α: *There exists an $A \subset V(G)$ such that*

- $|A| \geq \frac{n-k}{2} - \alpha n$, *and*
- $d(A) < \alpha$.

Extremal Case 2 (EC2) with parameter α**:** *There exists an* $A \subset V(G)$ *such that for* $B = V(G) \setminus A$ *we have*

- $\frac{n}{2} \geq |A| \geq \frac{n}{2} - \alpha n$, *and*
- $d(A, B) < \alpha$.

We say that a graph G is α-*extremal*, if one of these two cases holds, otherwise it is α-*non-extremal*. In the non-extremal case, one of the key tool is the following lemma on embedding cycles into almost balanced complete tripartite graphs.

Lemma 4 (Embedding into complete tripartite graphs)**.** *Let* G *be a complete tripartite graph with vertex sets* V_1, V_2, V_3*, where* $|V_i| = m_i$*,* $m_1 \leq m_2 \leq m_3$ *and* $m_1 \geq \frac{9}{10}m_3$*. If* $C_1, C_2, \ldots, C_s$ *is a collection of cycles, where* $|C_i| \geq 4$ *and* $\sum_{i=1}^{s} |C_i| = m_1 + m_2 + m_3$*, then* $C_1, C_2, \ldots, C_s$ *can be embedded into* G*.*

Proof. First we want to map $C_1, C_2, \ldots, C_s$ in such a way that after having mapped them, the size of the remaining vertex sets $V_1' \subset V_1$ and $V_2' \subset V_2$ differs only by at most 1. The balancing is done in the following way: First we map the cycles between V_1 and V_2 and also between V_1 and V_3. Of course we may have to map 1 point into the third vertex set. Once we have $||V_1|' - |V_2'|| \leq 1$ we continue the balancing. This time we map the cycles between V_1 and V_3 also between $V2$ and V_3. Because $m_1 \geq \frac{9}{10}m_3$ it is easy to see that we get a balancing $(||V_i'| - |V_j'|| \leq 1), 1 \leq i, j \leq 3$. Then we are done because in the later mappings we easily can keep the balance, and if for the last cycle C_s, $|C_s| = 3r + 2$ then the three remaining sets must have a size $r + 1, r + 1, r$. So we can map C_3 if $|C_s| = 3r + 1$ then the sizes are $r, r, r + 1$ and again we can easily map C_m. If $|C_m| = 3r$ entirely trivial. □

Finally in the extremal case (section 4.1) we will use the following simple simple facts on the sizes of a maximum set of vertex disjoint paths in G (see [6]).

Lemma 5. *In a graph* G *on* n *vertices, we have*

$$\nu_1(G) \geq \max\{\delta(G), \delta(G)\frac{n}{4\Delta(G)}\} \text{ and } \nu_2(G) \geq (\delta(G) - 1)\frac{n}{6\Delta(G)}$$

where $\nu_i(G)$ *denotes the size of maximum set of vertex disjoint paths of length* i *in* G*.*

3 The non-extremal case

Throughout this section we assume that we have a graph G satisfying (1) such that Extremal Cases 1 and 2 do not hold for G. We shall assume that n is sufficiently large and use the following main parameters

$$0 < \eta \ll \alpha \ll 1, \tag{2}$$

where $a \ll b$ means that a is sufficiently small compared to b.

Let $\gamma = k/n$, then we have $\delta(G) \geq (1 + \gamma)n/2$. For technical reasons we work with a slightly weaker minimum degree condition, we assume that

$$\delta(G) \geq \left(\frac{1 + \gamma}{2} - \eta\right) n. \tag{3}$$

In the non-extremal case this slightly smaller value of minimum degree is sufficient.

From Lemma 4 if G is an almost balanced complete tripartite graph then we can embed the cycles into G. Similarly if our graph G contains a union of 'big' complete tripartite graphs which cover almost all vertices of G then we can almost embed into G the cycles in H. But unfortunately this is not the situation in general, so we have to come up with a structure which is somewhat close to the tripartite graphs cover. For that purpose we are going to define the notion of a cover denoted by $\{\mathcal{T}, \mathcal{M}, \mathcal{I}\}$ where $\mathcal{T}$ is a collection of complete balanced tripartite graphs, $\mathcal{M}$ is a collection of balanced complete bipartite graphs with color classes of size $t = c\sqrt{\log n}$ and $\mathcal{I}$ is an almost independent vertex set ($e(\mathcal{I}) < \sqrt{\eta}n^2$).

3.1 The Optimal Cover

Let $\{\mathcal{T}, \mathcal{M}, \mathcal{I}\}$ be a cover, we denote by $V(\mathcal{T})$ the vertices in the tripartite graphs in $\mathcal{T}$ and $V(\mathcal{M})$ denotes the vertices in the bipartite graphs. We define the weight of the cover as follows:

$$2^{(\frac{1}{\eta})^2} \cdot |V(\mathcal{T})| \cdot 2^{\frac{1}{\eta}}|V(\mathcal{M})| \cdot c\sqrt{logn}$$

Let $\{\mathcal{T}, \mathcal{M}, \mathcal{I}\}$ be a cover of maximal weight, $\mathcal{T} = \{T_1, T_2, \ldots, T_p\}$ and $\mathcal{M} = \{M_1, M_2, \ldots, M_q\}$. Let $T_i = (V_1^i, V_2^i, V_3^i)$ and $M_i = (U_1^i, U_2^i)$. In the following we remark about some properties of such a maximal cover.

Claim 6. $|V(\mathcal{T})| \geq \alpha^4 n$.

This follows from Lemma 3 using (3) and the fact that we are not in Extremal Case 1. $\square$

Claim 7. *The number of triangles in $V(\mathcal{M}) \cup \mathcal{I}$ is at most $\eta^2 n^3$.*

Indeed, otherwise in $V(\mathcal{M}) \cup \mathcal{I}$, by Lemma 3 we could find some complete balanced tripartite graphs of size $c\sqrt{\log n}$, which results in a cover of larger weight. Note that an easy consequence of Claim 7 is that for most of the edges $e = (x, y) \in E(V(\mathcal{M}) \cup \mathcal{I})$, we have $|N(x, V(\mathcal{M}) \cup \mathcal{I}) \cap N(y, V(\mathcal{M}) \cup \mathcal{I})| \leq \eta n$. $\square$

Next we show that almost all vertices in $V(\mathcal{M}) \cup \mathcal{I}$ are densely connected to at most two color classes of almost all tripartite graphs in $\mathcal{T}$. Indeed, let $X \subset V(\mathcal{M}) \cup \mathcal{I}$ be set of those vertices that are densely connected to all three color classes of many tripartite graphs in $\mathcal{T}$. More precisely,

$$X = \{v \in V(\mathcal{M}) \cup \mathcal{I} \text{ ; there are at least } \eta p\, T_j\text{'s in } \mathcal{T}, \text{such that } |N(v, V_i^j)| \geq \eta |V_i^j|, 1 \leq i \leq 3\}$$

Claim 8. $|X| < \eta n$

Proof. Assume that $|X| \geq \eta n$, then clearly there exist at least $\eta^2 p/2$ tripartite graphs T_j such that for each such T_j there is a set $X_j^* \subset X$, such that $|X_j^*| \geq \eta^2 n/2$ and for each $v \in X_j^*$ we have $|N(v, V_i^j)| \geq \eta |V_i^j|$ for $1 \leq i \leq 3$. We will show that using vertices of X_j^* we can make some new tripartite graphs. We note that this type of argument will appear several times later, and at those places we will not go into the details again. Since $|V_1^j| = c\sqrt{logn}$, where c is a small constant, the number of subsets of $V(T_j)$ of size $3\eta|V_1^j|$ are at most $2^{3c\sqrt{\log n}} < \sqrt{n}$. Hence there are at least $\eta c\sqrt{\log n}$ vertices in X_j^* that have the same neighborhoods in all three color classes of T_j. Those vertices together with their neighborhoods in T_j make a complete 4-partite graph, which can be broken into four equal complete balanced tripartite graphs, while the remaining part of T_j is still a complete balanced tripartite graph. Therefore if we repeat the same process for all such T_j's, we increase the weight of the cover. Note that after this process we make all the tripartite and bipartite graphs of the same size, in the obvious way, hence all the color classes are still of the same size ($\frac{\eta c\sqrt{\log n}}{3}$) $\square$

Now we argue that almost all vertices in $V(\mathcal{M}) \cup \mathcal{I}$ are densely connected to at most one color class of almost all bipartite graphs in $\mathcal{M}$.

Claim 9. *There are at most ηn vertices $x \in V(\mathcal{M}) \cup \mathcal{I}$ such that $N(x, V(\mathcal{M}) \cup \mathcal{I})$ contains more than ηn^2 edges.*

This is any easy consequence of Claim 7, as otherwise there will be more triangles in $V(\mathcal{M}) \cup \mathcal{I}$. Note that this imply that for at most ηq bipartite graphs in $(U_1^j, U_2^j) \in \mathcal{M}$, there are more than ηn vertices $x \in V(\mathcal{M}) \cup \mathcal{I}$ such that $|N(x, U_i^j)| \geq \eta |U_i^j|, (1 \leq i \leq 2)$. □

3.2 The Structure of the Optimal Cover

After the above observations about the optimal cover, let us collect the structural information that we have about this optimal cover. Denote by $\tau = |V(\mathcal{T})|/3n$, $\mu = |V(\mathcal{M})|/2n$, $\beta = |\mathcal{I}|/n$ and recall that $\gamma = k/n$, where k is the number of odd cycles to be embedded.

From the fact that we are not in Extremal Case 1 we derive our main lemma of the non-extremal case.

Lemma 10. $\tau \geq \min(\gamma + \beta + \frac{\alpha^2}{2}, \frac{1}{3} - 2\eta)$.

Proof. We may assume that $\tau < 1/3 - 2\eta$, since otherwise we are done. Assume for contradiction that $\tau < \gamma + \beta + \alpha^2/2$, then either $\mu \geq \eta$ or $\beta \geq \eta$ (or maybe both). We distinguish two cases to prove this lemma based on the size of the independent set $\mathcal{I}$.

3.2.1 Case 1: There is an independent set ($\beta \geq \frac{\alpha^2}{2}$)

Let $\mathcal{I}' = \{x \in \mathcal{I} : N(x, \mathcal{I}) \geq \eta^3 |\mathcal{I}|$ or $|N(x, V(\mathcal{M}))| \geq (\mu + \eta)n$ or $|N(x, V(\mathcal{T}))| \geq (2\tau + \eta)n\}$. By definition of $\mathcal{I}$, Claim 9 and Claim 8, we have $|\mathcal{I}'| < 2\eta n$. We show that almost all vertices in $\mathcal{I}$ are almost completely connected to exactly two color classes of almost all tripartite graphs. Indeed, by (3) for every vertex $x \in \mathcal{I} \setminus \mathcal{I}'$ we have $|N(x, V(\mathcal{T}))| \geq \left(\frac{3\tau+\beta+\gamma}{2} - \eta\right) n$. Hence by the definition of $\mathcal{I}'$ and the assumption that $\tau < \gamma+\beta+\alpha^2/2$ for at least $(1-\alpha^2)t$ tripartite graphs $T_j \in \mathcal{T}$, we have that $|N(x, T_j)| \geq (2-\eta)|V_1^j|$.

Next we show that almost all vertices in $\mathcal{I} \setminus \mathcal{I}'$ have the almost the same neighborhood in $V(\mathcal{T})$. To see that for a tripartite graph T_j, let $\mathcal{I}_1 = \{x \in \mathcal{I} \setminus \mathcal{I}' : |N(x, T_j)| \geq (2-\eta)|V_1^j|$ and $|N(x, V_1^j)| \leq \eta |V_1^j|\}$. $\mathcal{I}_2$ and $\mathcal{I}_3$ are defined similarly. Now assume that two of these sets are large (say $|\mathcal{I}_1| \geq \eta n$ and $|\mathcal{I}_2| \geq \eta n$). But then by similar argument as above, we can make a complete tripartite graph of size $\eta |V_1^j|$, where the vertex sets are subsets of V_2^j, V_3^j and $\mathcal{I}_1$ respectively. Also, we can make another disjoint complete tripartite graph from the vertex sets V_1^j, V_3^j and $\mathcal{I}_2$. Note that the remaining part of T_j still has a complete tripartite graph of size $|V_1^j| - 2\eta |V_1^j|$ and a disjoint complete bipartite graph of size $\eta |V_1^j|$ (between arbitrary subsets of this size from V_1^j and V_2^j). In total the number of vertices in tripartite graphs remains the same but we get an additional bipartite graph. Therefore, if we have an ηp such T_j's then we can get another cover of larger weight. So we can assume that for almost all $x \in \mathcal{I} \setminus \mathcal{I}'$ we have $|N(x, T_j)| \geq (2-\eta)|V_1^j|$ and $|N(x, V_1^j)| \leq \eta |V_1^j|$ for at least $(1-\alpha^2)t$ T_j's. Hence we have that $d(\mathcal{I}, \bigcup_{j=1}^t V_1^j) < \eta$.

Furthermore we must have that $d(\bigcup_{j=1}^t V_1^j) < \eta$. Indeed otherwise by Lemma 2 we can make a collection of bipartite graphs in $\bigcup_{j=1}^t V_1^j$ (covering $\geq \eta^3 n$ vertices) and the vertices of those bipartite graphs can be replaced by some vertices of $\mathcal{I} \setminus \mathcal{I}'$ (as they are just the same as $\bigcup_{j=1}^t V_1^j$). This way again the size of tripartite graph cover remains the same but we have a larger bipartite graph cover hence larger weight.

With a very similar but simpler argument one can show that for almost all vertices $x \in \mathcal{I}$ and almost all $M_j \in \mathcal{M}$ we have $|N(x, U_2^j)| \geq (1-\eta)|U_2^j|$ and $|N(x, U_1^j)| \leq \eta |U_1^j|$. Furthermore we can show that $d(\bigcup_{j=1}^q U_1^j) < \eta$ and $d(\bigcup_{j=1}^q U_1^j, \bigcup_{j=1}^p V_1^j) < \eta$.

But then the set of vertices $(\mathcal{I} \cup \bigcup_{j=1}^p V_1^j \cup \bigcup_{j=1}^q U_1^j)$ is almost empty, while using the assumption $\tau < \gamma + \beta + \alpha^2/2$, its size is more than $(\frac{1-\gamma-\alpha}{2})n$, hence G is in Extremal Case 1, a contradiction.

3.2.2 Case 2: There is no independent set ($\beta < \frac{\alpha^2}{2}$ but $\mu > \alpha^2$)

First by Claim 8 we have that for almost all vertices $x \in V(\mathcal{M})$ we have that $|N(x, V(\mathcal{T}))| \leq (2\tau + \eta)n$. This together with (3) and the assumption that $\tau < \gamma + \alpha^2$ implies that $|N(x, V(\mathcal{M}))| \geq (\mu - \alpha^2)n$.

But by Claim 9 it is clear that almost all vertices in $V(\mathcal{M})$ are densely connected to at most one vertex class of almost all bipartite graphs in $\mathcal{M}$. Therefore by re-arranging we can assume that both $U_1 = \bigcup_{j=1}^q U_1^j)$ and $U_2 = \bigcup_{j=1}^q U_2^j)$ are almost empty and (U_1, U_2) is an almost complete bipartite graph (with density at least $(1-\alpha^2)$). Note that this also implies that $|N(x, V(\mathcal{T}))| \geq (2\tau - \alpha^2)n$.

Next we show that from almost all tripartite graphs in $\mathcal{T}$ at most one class is densely connected to both U_1 and U_2. More precisely, for $T_j \in \mathcal{T}$ let $\hat{V}_1^j = \{x \in V_1^j : x$ makes at least ηn^2 triangles with $E(U_1, U_2)\}$. $\hat{V}_2^j$ and $\hat{V}_3^j$ are defined similarly. Let $W_1 = \bigcup_{j=1}^p \hat{V}_1^j$, again W_2 and W_3 are similarly defined.

Now assume that two of these sets (say W_2 and W_3) are at least ηn. But then by Lemma 3 we can make complete tripartite graphs between in (W_2, U_1, U_2) and (W_3, U_1, U_2) that covers at least $\eta^3 n$ vertices. In these tripartite we just have to make sure that we use the same number of vertices from each V_2^j and V_3^j.

But then discarding some vertices from corresponding V_1^j we get a cover with larger weight, a contradiction. Therefore we must have that at least two of the sets (say W_1 and W_2) are empty. But then by the above density information, we may assume that (U_1, V_2), (U_2, V_1), (U_1, V_3) and (U_2, V_3) are all almost complete bipartite graphs, where $V_i = \bigcup_{j=1}^p V_1^j$, $(1 \leq i \leq 3)$.

It is easy to see that we must have that $d(U_1 \cup V_1) < \eta$. Indeed otherwise the graph induced by $(U_1 \cup V_1 \cup U_2)$ will have many triangles and hence tripartite graphs. Taking those tripartite graphs and replacing the vertices of V_1 in those by appropriate vertices of U_1, results in a cover of larger weight.

But with the assumption that $\tau < \gamma + \alpha^2$ we have that $|U_1 \cup V_1| \geq \frac{1+\gamma}{2} - \alpha$ and it have very few edges, hence G is in extremal case 1, a contradiction. This finishes the proof of Lemma 10.

3.3 The embedding algorithm

Given the optimal cover we are ready to describe the embedding procedure. Let us assume first that in Lemma 10 we have

$$\tau \geq \gamma + \beta + \alpha^2/2. \tag{4}$$

The other case when $\tau \geq 1/3 - \eta$ in Lemma 10 (the case of almost all triangles) is postponed until Section 3.4.

For notational convenience we assume that all color classes in the complete tripartite and bipartite graphs are of the same size l, where $l = O(\sqrt{\log n})$. Furthermore, we will assume in this section that in the cycle system H, all cycles are of length at most $\eta^2 l$, i.e. all cycles are small compared to the size of the color classes. We will give the embedding procedure for the case when some cycles are of length more than $\eta^2 l$ in Section 3.5.

After some preliminary embeddings in $\mathcal{T}$ we will embed cycles in such a way that we use up most of $\mathcal{M}$ and $\mathcal{I}$. The set of leftover vertices in $\mathcal{M}$ and $\mathcal{I}$ will be denoted by V_0 and we will call these exceptional vertices. We will embed cycles into the exceptional vertices as well, but for this purpose we might have to reembed a small portion of the cycles embedded earlier. We divide the embedding algorithm into phases according to this outline.

3.3.1 Phase 1: Preparatory embeddings in $\mathcal{T}$

In this phase we will embed some cycles in the complete tripartite graphs in $\mathcal{T}$. We start with some cycle embeddings that are called *exceptional* (as opposed to *typical*); these are the cycles that will help to embed the exceptional vertices. Given a complete tripartite graph $K_i^t = (V_1^{t,i}, V_2^{t,i}, V_3^{t,i})$ in $\mathcal{T}$ first we choose random subsets $U_j^{t,i} \subset V_j^{t,i}, j = 1, 2, 3$ of size $(\eta)^{1/3}l$.

Let us take the next unembedded cycle C_j in H of length at least 4 ((4) guarantees that there are still many such cycles). If $|C_j|$ is even, then we embed $|C_j|/2$ vertices arbitrarily into $U_1^{t,i}$ and $|C_j|/2$ vertices into $U_2^{t,i}$ (recall that we have complete bipartite graphs between the color classes). If $|C_j|$ is odd, then we embed $(|C_j| - 1)/2$ vertices into $U_1^{t,i}$, $(|C_j| - 1)/2$ vertices into $U_2^{t,i}$ and one vertex into $U_3^{t,i}$. For later cycles we embed them into the pairs $(U_2^{t,i}, U_3^{t,i})$ and $(U_1^{t,i}, U_3^{t,i})$ to make sure that we fill up the random subsets in the color classes in a balanced way. We continue the embedding of these exceptional cycles until most of the vertices are used up in these random subsets for all the complete tripartite graphs in $\mathcal{T}$.

Next we embed the triangles of H into $\mathcal{T}$ such that we embed almost the same number of vertices into each of the color classes in $\mathcal{T}$. Note that (4) implies that when we are done with Phase 1 we still have many unembedded cycles of length at least 4.

3.3.2 Phase 2: Handling atypical vertices

At this point first we will handle certain atypical vertices from $\mathcal{M} \cup \mathcal{I}$. A vertex $v \in \mathcal{M} \cup \mathcal{I}$ is called *atypical* if one (or more) of the following holds:

1. For a large portion ($\geq \eta$-portion) of the complete tripartite graphs $K_i^t = (V_1^{t,i}, V_2^{t,i}, V_3^{t,i})$ we have

$$deg(v, V_j^{t,i}) \geq \eta l, \; j = 1, 2, 3. \tag{5}$$

2. For a large portion ($\geq \eta$-portion) of the complete bipartite graphs $K_i^b = (V_1^{b,i}, V_2^{b,i})$ we have

$$deg(v, V_j^{b,i}) \geq \eta l, \; j = 1, 2. \tag{6}$$

3. $v \in \mathcal{I}$ and we have

$$deg(v, \mathcal{I}) \geq \sqrt{\eta}|\mathcal{I}|. \tag{7}$$

Note that the number of atypical vertices is small ($\leq \eta n$), since otherwise we could either increase the size of $\mathcal{T}$ or $\mathcal{M}$ by applying Lemma 10.

First we will handle the atypical vertices by embedding cycles into them. First we remove each atypical vertex $v \in \mathcal{M}$ from its complete bipartite graph. Then we remove some additional typical vertices to guarantee that we have the same number of typical vertices left in each complete bipartite graph. These vertices are added to $\mathcal{I}$. Furthermore, if we removed at least half of the vertices from a complete bipartite graph then we add all remaining typical vertices to $\mathcal{I}$.

Assume first that 1. holds for a $v \in \mathcal{M} \cup \mathcal{I}$. Take a complete tripartite graph $K_i^t = (V_1^{t,i}, V_2^{t,i}, V_3^{t,i})$ satisfying (5). Note that in this case by the Chernoff bound we may assume

$$deg(v, U_j^{t,i}) \geq \frac{\eta}{2}|U_j^{t,i}|, \ j = 1, 2, 3, \tag{8}$$

i.e. we have large neighborhoods in the random subsets as well. As suggested earlier we will "free up" the exceptional cycles embedded into K_i^t and we will reembed them in such a way that we use v. For this purpose first we reembed arbitrarily one exceptional cycle C in such a way that we use v and all other vertices come from $\cup_{j=1}^3 U_j^{t,i}$ ((8) guarantees that this is possible). Then we reembed the other exceptional cycles using Lemma 4. Indeed, we take the embedded vertices (in the earlier embedding) of the exceptional cycles other than C in $\cup_{j=1}^3 U_j^{t,i}$, we remove the vertices used in the reembedding of C and we add a few more vertices arbitrarily from K_i^t to make sure that the number of vertices available is the same as the sum of the lengths of the exceptional cycles other than C. Then by applying Lemma 4 we can reembed the other exceptional cycles since the difference in the sizes of the color classes is still small.

Assume next that 2. holds for a $v \in \mathcal{M} \cup \mathcal{I}$. Take a complete bipartite graph $K_i^b = (V_1^{b,i}, V_2^{b,i})$ satisfying (6). We embed a cycle C in such a way that we use v and all other vertices are from $\cup_{j=1}^2 V_j^{b,i}$. If we still have odd cycles then we select an odd cycle C, and thus we use the same number of vertices from the two color classes. If we have only even cycles left then from one of the color classes (select always the larger one) we use one more vertex. Thus at the end we have a discrepancy of size at most one between the sizes of the color classes so by removing at most one more typical vertex we can make the color classes equal in every complete bipartite graph.

Finally assume that 3. holds for a $v \in \mathcal{I}$ (note that these might be vertices added to $\mathcal{I}$ from $\mathcal{M}$). Let us take two typical vertices x and y in $N(v, \mathcal{I})$ (i.e. x and y do not satisfy 1., 2. or 3.; (7) guarantees that this is possible assuming $|\mathcal{I}| \geq \sqrt{\eta} n$). Then using the fact that x and y are typical, (3) and $3\tau + 2\mu + \beta = 1$ the pair (x, y) satisfies

$$deg(x, \mathcal{T}) + deg(y, \mathcal{T}) \geq \left(2\left(\frac{3\tau + 2\mu + \beta + \gamma}{2}\right) - 2\mu - \beta - \sqrt{\eta}\right) n = (3\tau + \gamma - \sqrt{\eta})n. \tag{9}$$

Thus we can select a complete tripartite graph $K_i^t = (V_1^{t,i}, V_2^{t,i}, V_3^{t,i})$ and an exceptional cycle C embedded into K_i^t such that we can reembed C by extending the path (x, v, y) inside $\cup_{j=1}^3 U_j^{t,i}$. The reembedding of the other exceptional cycles embedded into this K_i^t is similar to the case when 1. holds. Thus we may assume in the rest of the embedding algorithm that all remaining vertices are typical (i.e. they do not satisfy 1., 2. or 3.). For simplicity we still use the notation $\mathcal{M}$ and $\mathcal{I}$. This is the end of Phase 2.

3.3.3 Phase 3: Embedding into $\mathcal{M}$

If $\mu < \eta$, then we add the vertices in $\mathcal{M}$ to $\mathcal{I}$. Thus we may assume $\mu \geq \eta$. In this phase we will embed cycles into most of the vertices of $\mathcal{M}$. First we will embed the odd cycles and if we have no more odd cycles we move to the even cycles. Just as in Phase 1 first we will embed some exceptional cycles, the rest of the cycles will be called typical. As above, since all remaining vertices are typical, for every edge $e = (x, y)$ in the complete bipartite graphs in $\mathcal{M}$ we have (9).

Given a complete bipartite graph $K_i^b = (V_1^{b,i}, V_2^{b,i})$ in $\mathcal{M}$ again we choose random subsets $U_j^{b,i} \subset V_j^{b,i}, j = 1, 2$ of size $(\eta)^{1/3} l$. We embed the exceptional cycles in the following way. Let us take the next unembedded cycle C_j in H of length at least 4. If $|C_j|$ is even, then we embed $|C_j|/2$ vertices arbitrarily into $U_1^{b,i}$ and $|C_j|/2$ vertices into $U_2^{b,i}$. If $|C_j|$ is odd, first we select a random free edge (x, y) in $(U_1^{b,i}, U_2^{b,i})$ and a common free neighbor z of x and y in $\mathcal{T}$. It follows from (9) that z always exists. Furthermore, we may clearly select a z such that z is good for many edges ($\geq \eta$-portion) (x, y) of K_i^b (a fact that will be important later). Then C_j is embedded in the following way: one vertex is embedded into z, its two neighbors are embedded into x and y, half of the remaining $|C_j| - 3$ vertices are embedded arbitrarily into available free vertices in $U_1^{b,i}$ and the other half into $U_2^{b,i}$ (see Figure 1). We continue the embedding of these exceptional cycles until most of the vertices are used up in these random subsets for all the complete bipartite graphs in $\mathcal{M}$.

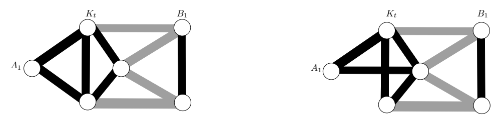

Figure 1: Embedding exceptional cycles in Phase 2 (on 5 vertices, where the rectangles indicate complete bipartite graphs)

Next we embed the typical cycles into $\mathcal{M}$. We start with the odd cycles. The embedding is similar to the exceptional odd cycles except we embed one vertex into each cluster of a complete tripartite graph to make sure that we fill up the clusters in $\mathcal{T}$ in a balanced way. So if $|C_j|$ is odd, first we select a random free edge (x, y) in a complete bipartite graph $K_{i_1}^b = (V_1^{b,i_1}, V_2^{b,i_1})$ in $\mathcal{M}$, a free neighbor z_1 of x in a cluster (say V_1^{t,i_2}) and a free neighbor z_2 of y in another cluster (say V_2^{t,i_2}) of a complete tripartite graph $K_{i_2}^t = (V_1^{t,i_2}, V_2^{t,i_2}, V_3^{t,i_2})$ in $\mathcal{T}$. Again it follows from (9) that z_1 and z_2 always exist. Furthermore, we may select z_1 and z_2 such that they are good for many edges (x, y) of $K_{i_1}^b$. Then C_j is embedded in the following way: 5 vertices of C_j are embedded into x, y, z_1, z_2 and an arbitrary $z_3 \in V_3^{t,i_2}$, half of the remaining $|C_j| - 5$ vertices (if there are any) are embedded arbitrarily into available free vertices in V_1^{b,i_1} and the other half into V_2^{b,i_1} (see Figure 2). When we run out of odd cycles we continue with the even cycles. For an even cycle C_j we just embed half of the vertices arbitrarily into available free vertices in V_1^{b,i_1} and the other half into V_2^{b,i_1}. We continue with the embedding process until most of the vertices are used up in $\mathcal{M}$.

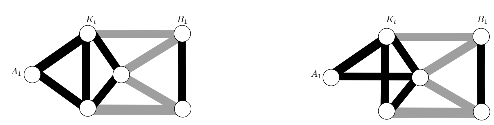

Figure 2: Embedding typical cycles in Phase 2 (on 5 vertices)

To show that we never get stuck, consider a situation when so far we embedded 3 vertices for k_i odd cycles (triangles in Phase 1 or typical cycles in this phase) into the complete tripartite graphs K_i^t, $1 \leq i \leq |\mathcal{T}|$, where

$$\sum_{i=1}^{|\mathcal{T}|} k_i < (\gamma - 2\sqrt{\eta})n$$

(i.e. we still have many unembedded odd cycles left). We will show that in this case we can embed one more odd cycle as above. Take a random edge (x, y) as in the above process. We will show that there must exist a complete tripartite graph K_i^t such that $k_i \leq (1 - \eta)l$ (call this K_i^t *not full*) and

$$deg(x, K_i^t) + deg(y, K_i^t) \geq 3l + k_i + \eta l. \tag{10}$$

Indeed, then we can select z_1 and z_2 from K_i^t as above since from each cluster in K_i^t we used $k_i \leq (1-\eta)l$ vertices so far and thus we can embed one more odd cycle into K_i^t. Assume indirectly that (10) does not hold for any of the complete tripartite graphs K_i^t that are not full. Then using this and the fact that x and y are typical we have

$$deg(x,\mathcal{T}) + deg(y,\mathcal{T}) < (3\tau + \sqrt{\eta})n + \sum_{i=1}^{|\mathcal{T}|} k_i < (3\tau + \gamma - \sqrt{\eta})n,$$

in contradiction with (9).

Thus either we use up most of $\mathcal{M}$ or we may continue with odd cycles until we have at most $2\sqrt{\eta}n$ unembedded odd cycles. If at this point we still have room in $\mathcal{M}$ then we continue with even cycles (note that in this case we have more than necessary even cycles from (4)). Thus either way we may use up all but η-portion of the vertices in $\mathcal{M}$. This is the end of Phase 3.

3.3.4 Phase 4: Embedding into $\mathcal{I}$

First we will embed cycles into most of $\mathcal{I}$. Since all the remaining vertices in $\mathcal{I}$ are typical we have the following for every $v \in \mathcal{I}$ using (3), (4) and $3\tau + 2\mu + \beta = 1$

$$deg(v,\mathcal{T}) \geq \left(\frac{3\tau + 2\mu + \beta + \gamma}{2} - \mu - \sqrt{\eta}\right)n = \left(\tau + \frac{\tau}{2} + \frac{\gamma+\beta}{2} - \sqrt{\eta}\right)n \geq \left(\tau + \gamma + \beta + \frac{\alpha^2}{4}\right)n. \quad (11)$$

Thus using (2) for many ($\geq \eta$-portion) complete tripartite graphs $K_i^t = (V_1^{t,i}, V_2^{t,i}, V_3^{t,i})$ we have

$$deg(v, K_i^t) \geq l + \left(\gamma + \beta + \frac{\alpha^2}{5}\right)\frac{n}{|\mathcal{T}|}. \quad (12)$$

This in turn implies that there are many ($\geq \eta/2$-portion) complete tripartite graphs K_i^t that are "good" for many ($\geq \eta/2$-portion) $v \in \mathcal{I}$ (i.e. v and K_i^t satisfy (12)). Let us take one such a complete tripartite graph $K_i^t = (V_1^{t,i}, V_2^{t,i}, V_3^{t,i})$ and the set of the $v \in \mathcal{I}$ that are good for it.

Let us consider the next unembedded cycle C, where $|C| = q, q \geq 4$. We distinguish the following cases depending on q.

Case 1: $q = 4r$, for some $r \geq 1$.
We select r vertices $v_1, \ldots, v_r \in \mathcal{I}$ that are good for K_i^t. Furthermore, using (12) we may select vertex disjoint paths P_i of length 2 centered at v_i, where the endpoints are free vertices from two different color classes of K_i^t. In fact we may assume that the endpoints always come from $V_1^{t,i}$ and $V_2^{t,i}$. Indeed, we may select $3r$ vertices and take the most common pair out of the three pairs. Put $\mathcal{I}_r = \{v_1, \ldots, v_r\}$. In order to embed C we think of $V_1^{t,i}, \mathcal{I}_r, V_2^{t,i}, V_3^{t,i}$ as a cycle of length 4 and we embed C by going around the cycle r times, where the embedded cycle contains all the P_i's as subpaths and the vertices from $V_3^{t,i}$ are arbitrary free vertices (see Figure 3).

Case 2: $q = 4r + 2$, for some $r \geq 1$.
Here we select $r+1$ vertices $v_1, \ldots, v_{r+1} \in \mathcal{I}$ that are good for K_i^t. For $v_1, \ldots, v_r$ again we find vertex disjoint paths P_i of length 2 centered at v_i and going to $V_1^{t,i}$ and $V_2^{t,i}$. For v_r and v_{r+1} we find two common free neighbors u_1 and u_2 from a common color class (say $V_2^{t,i}$) of K_i^t (again (12) makes this possible). To embed C we go around the cycle r times as in Case 1, but in the last cycle we "double up" on the $(\mathcal{I}_r, V_2^{t,i})$ edge, so embedding this part of the cycle to the subpath (v_r, u_1, v_{r+1}, u_2) and thus indeed using $4r + 2$ vertices.

Figure 3: Embedding C in Case 1 for $r = 1$

Then for the next cycle C' with $|C'| = 4r' + 2$ we double up on the other edge (i.e. $(V_1^{t,i}, V_3^{t,i})$) to make sure that we use up the vertices evenly from the color classes in K_i^t (see Figure 4).

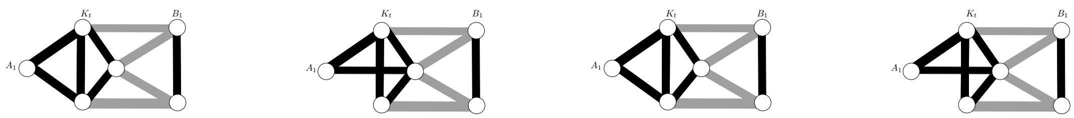

Figure 4: Embedding C and C' in Case 2 for $r = 1$

Case 3: $q = 4r + 1$, for some $r \geq 1$.
As above we select r vertices $v_1, \ldots, v_r \in \mathcal{I}$ that are good for K_i^t. For $v_1, \ldots, v_{r-1}$ (if $r > 1$) again we find vertex disjoint paths P_i of length 2 centered at v_i and going to $V_1^{t,i}$ and $V_2^{t,i}$. However, here for v_r we find a path P_r of length 2 (vertex disjoint from all the other paths) centered at v_r, where both endpoints come from $V_2^{t,i}$ (clearly this can be done from (12)). Then to embed C we go around the cycle r times but in the last cycle from $V_1^{t,i}$ instead of returning to $\mathcal{I}_r$ we double up in $V_2^{t,i}$ so we go to $V_2^{t,i}$ first and then close the cycle with the subpath P_r and thus indeed using $4r+1$ vertices. To eliminate the discrepancy caused by the doubling in $V_2^{t,i}$, in the next such cycle we go around the cycle $r - 1$ times (for $r > 1$) and then in the last cycle we skip $\mathcal{I}_r$ and double up in $(V_1^{t,i}, V_3^{t,i})$ (see Figure 5).

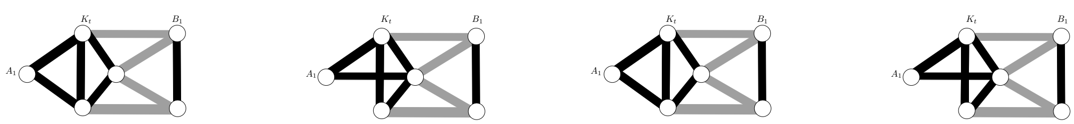

Figure 5: Embedding C and C' in Case 3 for $r = 1$

Case 4: $q = 4r + 3$, for some $r \geq 1$.
Similar to Case 3 but in the last cycle we double up in $V_2^{t,i}$ (as in Case 3) and in $(V_1^{t,i}, V_3^{t,i})$ at the same time to result in $4r + 3$ vertices and thus finishing all the cases (see Figure 6).

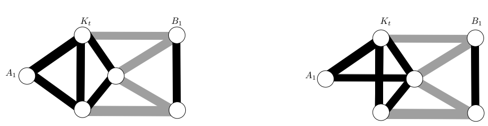

Figure 6: Embedding C in Case 4 for $r = 1$

To show again that we never get stuck, consider a situation when so far we embedded t_i vertices from $\mathcal{I}$ and k_i odd cycles (counting triangles in Phase 1, typical odd cycles in Phase 3 and odd cycles in this phase) with the help of a complete tripartite graph $K_i^t = (V_1^{t,i}, V_2^{t,i}, V_3^{t,i})$. Thus we have

$$\sum_{i=1}^{|\mathcal{T}|} k_i \le \gamma n \text{ and } \sum_{i=1}^{|\mathcal{T}|} t_i \le \beta n,$$

and we used up $t_i + k_i$ vertices from each cluster $V_j^{t,i}, j = 1, 2, 3$. Take the next $v \in \mathcal{I}$ in the above process. We will show that there must exist many ($\ge \eta$-portion) complete tripartite graphs K_i^t that are not full (so $t_i + k_i \le (1-\eta)l$) and

$$deg(v, K_i^t) \ge l + t_i + k_i + \frac{\alpha^2}{10} l. \tag{13}$$

Indeed, otherwise using the fact that v is typical

$$deg(v, \mathcal{T}) < \left(\tau + \frac{\alpha^2}{5}\right) n + \sum_{i=1}^{|\mathcal{T}|} t_i + \sum_{i=1}^{|\mathcal{T}|} k_i \le \left(\tau + \gamma + \beta + \frac{\alpha^2}{5}\right) n,$$

in contradiction with (11).

Then similarly as above we have many ($\ge \eta/2$-portion) complete tripartite graphs K_i^t that are not full and good for many $v \in \mathcal{I}$. Let us consider one such a complete tripartite graph K_i^t, the set of the $v \in \mathcal{I}$ that are good for it and the next unembedded cycle C, where $|C| = q, q \ge 4$. (13) implies that the above process goes through since for each cluster in K_i^t we used $t_i + k_i$ vertices so far, so indeed we can select the paths P_i as above.

We stop when we have exactly $\sqrt{\eta} n$ vertices (assume for simplicity that this is an integer) left in $\mathcal{I}$. If $\mathcal{I}$ is smaller than this to begin with then we add arbitrary vertices from $\mathcal{T}$ (unfortunately these might not be typical). Let us denote the set of these few remaining vertices by V_0. Thus we may assume that either all $v \in V_0$ are typical (so they do not satisfy 1., 2. or 3. above in Phase 2) or $\mathcal{I}$ is small. Furthermore, at the end of the procedure there are still only minor discrepancies in the sizes of the color classes in the complete tripartite graphs.

3.3.5 Phase 5: Handling vertices in V_0

We have $\sqrt{\eta} n$ vertices $v \in V_0$. For a vertex $v \in V_0$ we define R_v to be the set of vertices that may *replace* v, i.e. if a cycle is embedded using a vertex $x \in R_v$ then we can reembed the cycle using v and freeing up x and thus x may play indeed the role of v from now on. We will need the following claim.

Claim 11. *For every $v \in V_0$ we have*

$$|R_v| \ge \frac{1 - \gamma + \frac{\alpha^2}{4}}{2} n.$$

For this purpose first we will show that

$$|R_v \cap \mathcal{T}| \ge |N(v, \mathcal{T})| - \eta n, \tag{14}$$

and

$$|R_v \cap \mathcal{M}| \ge |N(v, \mathcal{M})| - \eta n. \tag{15}$$

In order, to prove (14) consider a complete tripartite graph $K_i^t = (V_1^{t,i}, V_2^{t,i}, V_3^{t,i})$ such that for some $1 \le j \le 3$ (say $j = 1$) we have

$$deg(v, V_j^{t,i}) \ge \eta l. \tag{16}$$

Then again

$$deg(v, U_j^{t,i}) \ge \frac{\eta}{2}|U_j^{t,i}|.$$

Consider an arbitrary $x \in V_2^{t,i}$, we will show that $x \in R_v$. Similarly as in Phase 2 we will reembed the exceptional cycles so that we will use v and free up x (if it is already used). For this purpose first we reembed an exceptional cycle assigned to the pair $(V_1^{t,i}, V_2^{t,i})$ such that we use v (and do not use x) and all other vertices come from $\cup_{j=1}^3 U_j^{t,i}$. Now if x is already used in an embedding of a cycle (exceptional, or cycles used for $\mathcal{M}$ and $\mathcal{I}$), then we can clearly replace x with a vertex from $U_2^{t,i}$ and reembed the cycle using the fact that $U_2^{t,i}$ is a random subset of $V_2^{t,i}$ so if a vertex has many neighbors in $V_2^{t,i}$ then it also has many neighbors in $U_2^{t,i}$. The remaining exceptional cycles can be reembedded by using Lemma 4 as in Phase 3 such that we do not use x. Thus v plays the role of a vertex from $V_2^{t,i}$, x is freed up and may play the role of v. Similarly, we can show that for any $x \in V_3^{t,i}$ we have $x \in R_v$. Finally by Lemma 4 if we can bring out any vertex of $V_2^{t,i}$ to play the role of v, then instead we can bring out any vertex of $V_1^{t,i}$; this should not make a difference since the overall number of remaining vertices is the same. Thus if we have (16) for v and a K_i^t, then $K_i^t \subset R_v$ and this clearly implies (14).

To prove (15) consider a complete bipartite graph $K_i^b = (V_1^{b,i}, V_2^{b,i})$ such that for some $1 \le j \le 2$ (say $j = 1$) we have

$$deg(v, V_j^{b,i}) \ge \eta l. \tag{17}$$

Then again

$$deg(v, U_j^{b,i}) \ge \frac{\eta}{2}|U_j^{b,i}|.$$

Consider an arbitrary $x \in V_2^{b,i}$, we will show that for most of these vertices we have $x \in R_v$. Similarly as above we will reembed the exceptional cycles so that we will use v and free up x. First we reembed an exceptional cycle C such that we use v and all other vertices (except the bridge vertex z defined in Phase 2) come from $\cup_{j=1}^2 U_j^{b,i}$. For this purpose we consider a bridge and a non-bridge neighbor of v (these exist) along with the bridge and extend it to a reembedding of C. Next consider the cycle C' whose embedding is using x. In this embedding of C' we replace x with a vertex x' from $U_2^{b,i}$. The only difficulty is when x is a bridge vertex. Assume thus that x is bridge vertex. Since the bridges were chosen so that z (in the exceptional case) and z_1, z_2 (in the typical case, see Phase 3) are good not just for this edge but for many other edges as well in $(V_1^{b,i}, V_2^{b,i})$, thus z or z_2 has many neighbors in $U_2^{b,i}$ and we can pick one of them as x'. The remaining exceptional cycles can easily be reembedded. Thus if we have (17) for v and a K_i^b then all vertices of $V_2^{b,i}$ can be put in R_v and this clearly implies (15).

We also have

$$|N(v, \mathcal{I})| \le \sqrt{\eta} n, \tag{18}$$

since by the above either all $v \in V_0$ are typical or $\mathcal{I}$ is small ($|\mathcal{I}| \le \sqrt{\eta} n$). Then (14), (15) and (18) imply together that

$$|R_v| \ge deg(v) - 2\sqrt{\eta} n \ge \frac{1 + \gamma - 4\sqrt{\eta}}{2} n.$$

This indeed implies Claim 11 if $\gamma \geq \frac{\alpha^2}{4}$. Assume $\gamma < \frac{\alpha^2}{4}$. Consider $N(R_v) = \cup_{x \in R_v} N(x)$. If

$$|N(R_v)| \geq \frac{1 - \gamma + \frac{\alpha^2}{2}}{2} n,$$

then again we have Claim 11 as above, since any vertex $x \in R_v$ may play the role of v. Thus we may assume

$$\frac{1 - \gamma - 4\sqrt{\eta}}{2} n \leq |R_v|, |N(R_v)| \leq \frac{1 - \gamma + \frac{\alpha^2}{2}}{2} n,$$

where $\gamma < \frac{\alpha^2}{4}$. Then considering $A = R_v$ and $B = (R_v \setminus N(R_v)) \cup (V(G) \setminus (R_v \cup N(R_v))$, we have no edges between A and B by definiton. This is in contradiction with the fact that we are not in Extremal Case 1 (if A and B are roughly the same) and we are not in Extremal Case 2 (if A and B are almost disjoint). Note that the intermediate case gives a contradiction immediately as the neighbors of $R_v \setminus N(R_v)$ can only go to $N(R_v) \setminus R_v$. Thus we may assume that Claim 11 is true.

Now we are ready to finish the embedding algorithm. We assign unembedded cycles to the remainder of $K_1^t = (V_1^{t,1}, V_2^{t,1}, V_3^{t,1})$ until we run out of room, i.e. $r > 0$ vertices are missing for the next cycle. To make up for this deficiency we assign r vertices from V_0 to K_1^t. Claim 11 and (3) guarantee that we can replace these r vertices with r vertices that all have large degree to $V_1^{t,1}$. By reembedding exceptional cycles we can use up these added vertices. In the remainder of K_1^t the assignment is perfect, i.e. the number of remaining vertices is equal to the sum of the lengths of the remaining assigned cycles, the difference between the sizes of the color classes is still small, and thus Lemma 4 finishes the embedding.

We continue in this fashion for all the complete tripartite graphs. Finally we assign all remaining vertices of V_0 to the last complete tripartite graph $K_{|\mathcal{T}|}^t$. Since overall the sum of the cycle lengths is equal to the total number of vertices, the assignment will be perfect in $K_{|\mathcal{T}|}^t$. This finishes the embedding algorithm. Note the new way how we handled the remaining few vertices in V_0, essentially they could be placed anywhere, making a usually difficult task easy. This construction could be interesting on its own.

3.4 The case of almost all triangles

Note that this case might be interesting on its own as it can be viewed as a generalization of the Corrádi-Hajnal theorem [9]. We have $\tau \geq 1/3 - \eta$ in Lemma 10 so the cover consists almost entirely of complete tripartite graphs. If there are not too many odd cycles in the cycle system H, say $\gamma < 1/3 - 4\eta - \alpha^2/2$, then considering the set of vertices that are not covered by the tripartite graphs as our independent set $\mathcal{I}$, so $\beta \leq 3\eta$, the other inequality in Lemma 10 is also satisfied, since $\tau \geq 1/3 - \eta \geq \gamma + \beta + \alpha^2/2$, hence we can apply the previous embedding procedure. Therefore, we may assume that there are many odd cycles, $\gamma \geq 1/3 - 4\eta - \alpha^2/2$, which together with (3) and (2) imply that $\delta(G) \geq (2/3 - \alpha^2)n$. Furthermore, it follows that the H contains at least $(1/3 - 3\alpha^2)n$ triangles. Indeed, otherwise the total number of vertices is at least

$$3(1/3 - 3\alpha^2)n + 5(2\alpha^2)n = n + \alpha^2 n > n,$$

a contradiction.

In the optimal cover we have at most $3\eta n$ points outside the complete tripartite graphs $\mathcal{T}$. By greedily embedding the non-triangles into the complete tripartite graphs (such that we do not embed too many vertices into each complete tripartite graph and that we keep the balance inside each tripartite graph) we may assume that we have only triangles left in H and thus the number of vertices outside the complete tripartite graphs is divisible by three. We will consider only three vertices outside $\mathcal{T}$ and extend the cover by one or more triangles to include these three vertices, such that the cover remains a balanced one. By repeating this procedure we eliminate all the vertices outside the complete tripartite graphs and then the remaining triangles of H can be embedded inside the complete tripartite graphs. Therefore, we consider only three vertices a, b and c outside $\mathcal{T}$ which do not make a triangle.

For $i \in \{1, 2, 3\}$ we say that a vertex v is *i-sided* to a tripartite graph $K_t = (V_1, V_2, V_3)$ if we have $d(v, K_t) \geq ((i-1)/3 + \eta)$, i.e. v has a large degree to at least i color classes. Denote by $s(v, K_t)$ the largest i for which v is i-sided to K_t.

Following a similar approach as in Phase 5 above, if v is two-sided to K_t (say to the pair (V_2, V_3)) then we say that the vertices in V_1 may *replace* v, i.e. any of the vertices in V_1 can be exchanged with v while keeping the cover balanced. Similarly, if v is three-sided to K_t then the vertices of all three color classes may replace v. For a vertex v, define R_v to be the set of vertices that may replace v over all tripartite graphs. By the minimum degree condition, for any vertex v we have $|R_v| \geq (1/3 - \alpha)n$. Furthermore, whenever a vertex v is exchanged with a vertex of a complete tripartite graph K_t, then we immediately cover it with a triangle in K_t to maintain the property that we still have a balanced complete tripartite graph. Note also that if there exists a $K_t \in \mathcal{T}$ such that a, b and c are two-sided to 3 different pairs of color classes in K_t (this happens for example if a, b and c are all three-sided to K_t), then we can easily expand the cover by three triangles such that we eliminate a, b and c and we keep the balance inside K_t (see Figure 7) so we may assume that this is never the case.

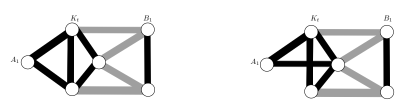

Figure 7: a, b and c are 2-sided to different pairs

If $s(v, K_i) \leq 1$ for an α-fraction of the K_i's in $\mathcal{T}$, then there is at least an $\frac{\alpha}{2}$-fraction of the other K_j's in $\mathcal{T}$ such that $s(v, K_j) = 3$. However, then from each such K_j all three color classes are exchangeable with v, hence $|R_v| > (1+\alpha)n/3$. So R_v is small (smaller than $(1+\alpha)n/3$) only if v is two-sided to most of the tripartite graphs. Furthermore, since any vertex in R_v may play the role of v, R_v is small only if every $x \in R_v$ is two-sided to the same pair of clusters in most of the tripartite graphs. This implies that we have an almost complete bipartite graph between R_v and its complement and this extremal case can be handled the same way as Extremal Case 1.

Thus we may assume that for any vertex v (in particular for a, b and c) we have $|R_v| > (1+\alpha)n/3$. This implies that for two of the vertices a, b and c (say for a and b) we have $|R_a \cap R_b| \geq \alpha n/3$. In this case we say that a and b *collapse*. Indeed, let $R_a \cap R_b = \{x_1, \ldots, x_t\}$, where $t \geq \alpha n/3$. Then there is a set $R_{a,b} \subset \cup_{x_i} R_{x_i}$, such that $|R_{a,b}| \geq (1 + \frac{\alpha}{2})n/3$ and for all $v \in R_{a,b}$, we have $v \in R_{x_i}$ for at least two vertices x_i. We call this process the *expansion process.* Then any two vertices of $R_{a,b}$ may play the role of a and b.

Thus we may assume

$$|R_{a,b}|, |R_c| \geq \left(1 + \frac{\alpha}{2}\right) \frac{n}{3}.$$

Then the minimum degree condition implies that for any vertex v

$$deg(v, R_{a,b}), deg(v, R_c) \geq \frac{\alpha}{7} n.$$

From this it follows using Fact **??** that for any color class V of any complete tripartite graph there exist at least $\alpha n/14$ vertices in $R_{a,b}$ (and in R_c) with at least $\alpha|V|/14$ neighbors in V, a fact that will be important later.

We will show that either a, b and c collapse or we can extend the cover to include a, b and c, as desired. Thus assume first that a, b and c cannot be collapsed.

Consider a vertex $x \in R_c$ and a $K_t \in \mathcal{T}$. There is at most one vertex $y \in R_{a,b}$ such that $d(x, K_t) + d(y, K_t) \geq (4/3 + \eta)$, because otherwise either a, b and c collapse or we have three vertices that are two-sided to three different pairs. This and the minimum degree condition imply that for most ($\geq (1-\alpha^2)$-fraction) of the tripartite graphs K_t (call these *typical*) for most ($\geq (1-\alpha^2)$-fraction) of the vertices $y \in R_{a,b}$ we have the following density condition

$$(4/3 - \alpha^2) \leq d(x, K_t) + d(y, K_t) \leq (4/3 + \eta). \tag{19}$$

Classify these typical complete tripartite graphs K_t in the following way: K_t is type j if $s(x, K_t) = j, 1 \leq j \leq 3$.

Assume first that a type 2 typical K_t exists. Since $s(x, K_t) = 2$, we have $d(x, K_t) < \left(\frac{2}{3} + \eta\right)$, and thus from (19) for most vertices $y \in R_{a,b}$ we get $d(y, K_t) \geq \left(\frac{2}{3} - 2\alpha^2\right)$. This implies that most ($\geq (1-2\alpha^2)$-fraction) vertices $y \in R_{a,b}$ are two-sided to the same pair of clusters (say (V_1, V_2)) in K_t and this pair is different from the pair of clusters to which x is two-sided, say (V_1, V_3). But then by the above remark we have many ($\geq \alpha n/15$) vertices $y \in R_{a,b}$ which have many neighbors in V_3 as well, and thus these vertices are three-sided to K_t, and thus we can extend the cover by three triangles, as desired.

Thus we may assume that we have no type 2 K_t, i.e. all typical complete tripartite graphs are of type 1 or 3. Consider a type 3 $K_t \in \mathcal{T}$. We can have at most one vertex $y \in R_{a,b}$ that is at least two-sided to K_t, since otherwise again a, b and c collapse or we have three vertices to three different pairs. This and (19) imply that x is almost complete to K_t and most vertices $y \in R_{a,b}$ are almost complete to one color class in K_t. Similarly, this is true symmetrically for a type 1 K_t, x is almost complete to one color class of K_t and most vertices $y \in R_{a,b}$ are almost complete to K_t. In particular, this implies that the number of type 1 K_t's is roughly the same as the number of type 3 K_t's. Furthermore, this implies that all other vertices $x' \in R_c$ are almost complete to one color class of a type 1 K_t (otherwise we are done again) and almost complete to a type 3 K_t. Hence the union of type 1 K_t's is $R_{a,b}$, the union of type 3 K_t's is R_c, $R_{a,b}$ and R_c are almost complete and roughly they have the same size. Then consider an arbitrary $x \in R_c$ and an edge (y_1, y_2) within the set $N(x, R_{a,b})$ (this must exist as $R_{a,b}$ is almost complete and any $x \in R_c$ is connected to about one third of $R_{a,b}$), this forms a triangle (y_1, y_2, x) replacing a, b and c and this extends our triangle cover, as desired.

Finally, let us assume that a, b and c collapse, i.e. R_c and R_{ab} are not disjoint.

As above, using the expansion process we may assume $|R_{a,b,c}| > (1 + \frac{\alpha}{2})n/3$. We will show that actually $|R_{a,b,c}| \geq (1-\alpha)2n/3$. Indeed, otherwise there is an α-fraction of tripartite graphs, $\mathcal{T}' \subset \mathcal{T}$, where only one color class (say V_3) is part of $R_{a,b,c}$. Then most of the vertices in $R_{a,b,c}$ must be two-sided to (V_1, V_2) in most of the tripartite graphs in $\mathcal{T}'$ (since otherwise we would get three vertices that are three-sided to the same K_t). But then again by the above remark we have many vertices $y \in R_{a,b,c}$ which have many neighbors in V_3 as well, and thus these vertices are three-sided to K_t, and thus we can extend the cover by three triangles, as desired. Now if $|R_{a,b,c}| \geq (1-\alpha)2n/3$, then for any vertex $x \in R_{a,b,c}$, $|N(x) \cap R_{a,b,c})| \geq (1-\alpha)n/3$. By non-extremality there are edges inside $N(x) \cap R_{a,b,c}$ which form triangles with x. We are done since we found a triangle in $R_{a,b,c}$ and those three vertices can replace a, b and c. This finishes the proof of the case of almost all triangles.

3.5 Embedding Long Cycles

In this case we have some cycles longer than $\eta^2 l = c\sqrt{\log n}$. It is not hard to see that all the arguments in the previous section in the non-extremal case work even when the minimum degree is slightly less, i.e if we have $\delta(G) \geq (\frac{1+\gamma}{2} - \eta)n$.

We use the following standard lemma for randomly splitting a graph into two subgraphs such that the relative minimum degree and non-extremality in the two subgraphs is roughly the same as in the original graph.

Lemma 12. *For any $0 < \varepsilon < 1$, there exists an n_0, such that if H is an α-non-extremal graph on $n \geq n_0$ vertices with $\delta(H) \geq \lambda n$ then for any random subset A of $V(H)$, with $\varepsilon n \leq |A| \leq (1-\varepsilon)n$, (let $B = V(G) \setminus A$) we have with high probability that $\delta(H|_A) \geq (\lambda - n^{-\frac{1}{3}})|A|$, $\delta(H|_B) \geq (\lambda - n^{-\frac{1}{3}})|B|$ and both $H|_A$ and $H|_B$ are $(\alpha - n^{-\frac{1}{3}})$-non-extremal.*

The first half of the statement on the minimum degrees can be found in [3] (Lemma 2.3). The second half of the statement on the non-extremality can also be proved by a standard argument. Indeed, being extremal is equivalent to the fact that for most pairs of vertices x, y the size of the common neighborhood $N(x) \cap N(y)$ is either around $n/2$ or 0. Thus if the graph H is non-extremal then we have many pairs for which this is not true. However, this fact in inherited by the random subgraphs $H|_A$ and $H|_B$.

Furthermore we will make use of the following simple fact.

Fact 13 ([10])**.** *Every 2-connected graph H on n vertices has a cycle of length* $\min\{n, 2\delta(H)\}$.

Using the fact that our graph G is α-non-extremal we prove the following extensions of Dirac's theorem [10] on Hamiltonian graphs and Bondy's theorem [8] on pancyclic graphs. Both of these are folklore, but for the sake of completeness we sketch the proofs.

Lemma 14. *For every $\alpha > 0$ there exist constants $\eta, n_0 > 0$ such that if H is an α-non-extremal graph on $n \geq n_0$ vertices with $\delta(H) \geq (1/2 - \eta)n$, then H is Hamiltonian.*

Proof. Using Fact 13 (note that H is clearly 2-connected as it is not in Extremal Case 2) we get a cycle $C = u_1, \ldots, u_q$; $q \geq (1 - 2\eta)n$. If $q < n$, we will insert the vertices outside C to extend the cycle. Let a be an outside vertex. If a is connected to u_i, u_{i+1}, then we can insert a between u_i and u_{i+1} to extend C. If not, then by the minimum degree condition, a must be connected to many pairs u_{i-1}, u_{i+1}. Let $R_a = \{u_i \in C : u_{i-1}, u_{i+1} \in N(a)\}$. Then we have $|R_a| \geq (1/2 - 4\eta)n$. Since G is not α-extremal, $d(R_a) \geq \alpha$. Consider an edge (u_j, u_k) inside R_a. By definition a is connected to $u_{j-1}, u_{j+1}, u_{k-1}, u_{k+1}$. Since there are many such edges we may assume $k > j + 5$. Then the following cycle extends C: $u_{j+1}, u_{j+2} \ldots, u_{k-1}, u_k, u_j, u_{j-1}, u_{j-2}, \ldots, u_{k+1}, a, u_{j+1}$ (see Figure 8). $\square$

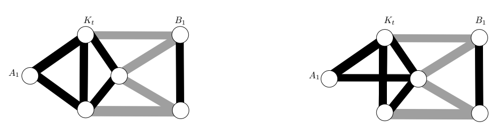

Figure 8: The dashed-cycle extends C to include a

We now prove the following stronger statement.

Lemma 15. *For every $\alpha > 0$ there exist constants $\eta, n_0 > 0$ such that if H is an α-non-extremal graph on $n \geq n_0$ vertices with $\delta(H) \geq (1/2 - \eta)n$, then H is pancyclic, i.e. H has a cycle of length q for all $3 \leq q \leq n$.*

Proof. For $q \geq \varepsilon n$ $(0 < \varepsilon < \eta)$, we randomly choose a subset $A \subset V(H)$, $|A| = q$. By Lemma 12, $H|_A$ satisfies the conditions of Lemma 14, hence we are done. For $q < \varepsilon n$, we choose a path, $P = v_2, \ldots, v_{q-1}$ on $(q-2)$ vertices (such a path obviously exits, as H is Hamiltonian). $N(v_2)$ and $N(v_{q-1})$ in $V(H) \setminus V(P)$ both have size at least $(1/2 - 2\eta)n$. Hence by α-non-extremality there are edges between $N(v_2)$ and $N(v_{q-1})$. Take one such edge (v_1, v_q) and $v_1, v_2, \ldots, v_{q-1}, v_q$ is a cycle on q vertices. □

Now we prove our main theorem when the cycle system has longer cycles, i.e. at least one cycle is longer than $\eta^2 l = c\sqrt{\log n} = m$. We denote by H the given cycle system and by H_s the set of smaller cycles in H (cycles of length at most m) and let $H_l = H \setminus H_s$. Let M be the total number of vertices in cycles in H_s. We consider the following cases depending on M.

Case 1: $0 < M < \eta n$.
In this case we embed the cycles in H_s one by one using Lemma 15. The remaining graph still has minimum degree at least $(\frac{1+\gamma}{2} - 2\eta)(n - M)$. To embed H_l in the remaining subgraph we get Case 4.

Case 2: $\eta n \leq M \leq (1-\eta)n$.
In this case we randomly partition $V(G)$ into A and B, $|A| = M$. By Lemma 12 we have $\delta(G|_A) \geq (\frac{1+\gamma}{2} - n^{-\frac{1}{3}})|A|$ and $\delta(G|_B) \geq (\frac{1+\gamma}{2} - n^{-\frac{1}{3}})|B|$ and both $G|_A$ and $G|_B$ are $(\alpha - n^{-\frac{1}{3}})$-non-extremal. We embed the cycles in H_s in $G|_A$ applying the procedure in the previous section and for embedding H_l in $G|_B$ we get Case 4.

Case 3: $M > (1-\eta)n$.
Similarly as in Case 1, we embed the cycles in H_l one by one using Lemma 15. The remaining graph still has minimum degree at least $(\frac{1+\gamma}{2} - 2\eta)M$ and is $(\alpha/2)$-non-extremal, so we use the procedure in the previous section to embed the cycles in H_s in the remaining subgraph as now all cycles are small.

Case 4: $M = 0$, i.e. all cycles are of length at least m.
So we have a graph G on n vertices that is α-non-extremal, we have $\delta(G) \geq (1/2 - \eta)n$ and let H be the given cycle system. Let the cycles be $C_1, C_2, \ldots, C_r$ with length $n_1 \geq n_2 \geq \ldots \geq n_r$, where $n_i \geq m$ for all $1 \leq i \leq r$. We follow a similar approach as in [3] (Section 5). We consider two cases based on the value of n_1.

Case 4.1: $n_1 \geq (1-\eta)n$.
We embed all cycles C_i for $i > 1$ one by one using Lemma 15. It is easy to see that the remaining graph satisfies the conditions of Lemma 14, hence there is a Hamiltonian cycle C_1 in it.

Case 4.2: $n_i < (1-\eta)n$ for $1 \leq i \leq r$.
We distribute the cycles into two sets H_1 and H_2, such that if n_A and n_B denote the total number of vertices in cycles in H_1 and H_2, then we have $n_A, n_B \leq (1-\eta)n$. We randomly partition $V(G)$ into two sets A and B such that $|A| = n_A$ and $|B| = n_B$. We will embed the cycles in H_1 and H_2 into $G|_A$ and $G|_B$, respectively.

We recursively apply the above splitting procedure until the condition of Case 4.1 is satisfied in all the parts. We will show that the minimum degree and non-extremality conditions hold until the end of this process, hence the required cycles can be found in each part as in Case 4.1. Define the normalized degree of a graph F as $D(F) = \frac{\delta(F)}{|V(F)|}$. Initially we have $D(G) \geq (1/2 - \eta)$, therefore by Lemma 12 we have

$$D(G_A) \geq D(G) - n^{-\frac{1}{3}} \quad \text{and} \quad D(G_B) \geq D(G) - n^{-\frac{1}{3}}.$$

Since the splitting process terminates with each part of size at least m and each time the number of vertices is reduced by at least an η-factor, for each final subgraph G_f we get

$$D(G_f) \geq D(G) - m^{-\frac{1}{3}} \sum_{i=0}^{\infty} (1-\eta)^{i/3} \geq \frac{1}{2} - 2\eta,$$

using the fact that n (and thus m) is sufficiently large. A similar computation shows that each final G_f is $\alpha/2$-non-extremal, hence the conditions of Lemma 15 are satisfied so we can apply the procedure of Case 4.1 in each part, finishing the proof in this case.

Let us assume that $G = (V(G), E(G))$, $|V(G) = n|$, $|E(G)| \geq \frac{1}{4}n^2$ and G is not α extremal. Notice that if G is not α extremal, the there are at least αn^2 $\{x, y\}$ pairs, $x, y \in V(G)$ such that $|N(x) \cap N(y)| > \alpha n$ and $|N(x) \cap N(y) - \frac{1}{2}n| > \alpha n$. We call these pairs good pairs. Now we are going to apply the same partiontioning algorithm. What we have to maintain is the proper number of good pairs. The size of the intersection of the neiborhoods of good pairs can be controlled the same way as the degree of vertices was controlled. So the real thing is to control the proper number of good pairs. But that is not difficult either. Form a graph where the edges are good pairs. We think this graph as a union of "big" matching. If we partition a graph G with vertex set A so that we choose a point of A with probability α, where say $\alpha > |A|^{-1/4}$, then if the size of the matching is bigger than say $|A|^{7/8}$, in the matching the right number of edges will be A' and compliment of A', where $|A'| = \alpha|A|$ and $|A'| = (1-\alpha)|A|$. ($\alpha > |A|^{-1/4}$ unless among the cycles which we want to map in A the largest cycle is $> |A| - |A|^{3/4}$. But then we use Lemma 19 to embed the cycles.) Here we use that the matching edges are independent. That is why we consider a graph as a union of matchings.

Take one of the matchings, if we randomly partition the vertices of the graph into almost equal part, then with exponentially high probabilities that the number of edges of the chosen matching will be $\frac{1}{4}m \pm m^{\frac{3}{4}}$ where m is the size of the matching. Continuing this way, we can then maintain the proper number of pairs until the very end of the procedure.

4 The Extremal Cases

In the extremal cases we will repeatedly use the following simple fact.

Fact 16. *If $G(A, B)$ is a bipartite graph, with*

- *$deg(a, B) \geq (1-\eta)|B|$ for all $a \in A$,*
- *$deg(b, A) \geq (1-\eta)|A|$ for all $b \in B$,*
- *B has a matching M of size k and $|B| = |A| + k$,*

then if H is a set of cycles, such that the sum of the lengths of the cycles is $|A| + |B|$ and the number of odd cycles is k, then H can be embedded into G.

The basic idea is to use one edge of M for each odd cycle and all other edges of the cycles will be found in the almost complete bipartite graph between A and B. We will assign $\lceil\frac{|C|}{2}\rceil$ vertices of B to each cycle C in H. To the next cycle $C_i \in H$ with $|C_i| = 2s$, we assign s unassigned vertices $x_i^1, x_i^2, \ldots, x_i^s$ in B that are disjoint from M. For C_i with $|C_i| = 2s+1$, we assign $s-1$ unassigned vertices $x_i^1, x_i^2, \ldots, x_i^{s-1}$ disjoint from M and an unassigned edge $e_i = (y_i^1, y_i^2)$ from M. Next we will define an auxiliary bipartite graph $G'(A, B')$ in the following way: B' has a vertex corresponding to successive pairs of vertices assigned to C_i, i.e. $(x_i^1, x_i^2), (x_i^2, x_i^3), \ldots, (x_i^s, x_i^1)$ if $|C_i| = 2s$. For each odd cycles C_i, B' has a vertex for pairs $(y_i^1, x_i^1), (x_i^1, x_i^2), (x_i^2, x_i^3), \ldots, (x_i^{s-1}, y_i^2)$. In G', every vertex in B' is connected to common neighbors of the vertices in the corresponding pair in G.

Clearly we have $|A| = |B'|$ and the minimum degree of a vertex in G' is at least $(1-2\eta)|A|$. Hence by the König-Hall theorem G' has a perfect matching. It is easy to see that any perfect matching in G' corresponds to an embedding of H in G. $\square$

4.1 Extremal Case 1

Here our graph G satisfies (1) and we are in Extremal Case 1.

Extremal Case 1 (EC1) with parameter α**:** *There exists an* $A \subset V(G)$ *such that*

- $|A| \geq \frac{n-k}{2} - \alpha n$*, and*
- $d(A) < \alpha$*.*

By adding or deleting vertices to or from A we may achieve that $|A| = (n-k)/2$ and $|B| = (n+k)/2$ (note that these are always integers). Furthermore, an easy computation shows that we still have $d(A) < 10\alpha$ (for simplicity we keep the notation A, B). This and (1) imply that we have

$$d(A, B) > 1 - 10\alpha. \tag{20}$$

Thus roughly speaking, we have an almost complete bipartite graph between A and B. The basic idea is to find a matching of size k (using (1) and Lemma 5 as we have $\delta(G|_B) \geq k$) in B and then use Fact 16 to embed the cycles. However, we have to deal with certain exceptional vertices first.

A vertex $v \in A$ (similarly in B) is called *exceptional* if it is *not* connected to most of the vertices in the other set, more precisely if we have

$$deg(v, B) \leq (1 - \sqrt{10\alpha})|B|.$$

Let us denote the set of exceptional vertices by E_A in A and by E_B in B. From (20) we get that we have few exceptional vertices

$$|E_A| \leq \sqrt{10\alpha}|A| \text{ and } |E_B| \leq \sqrt{10\alpha}|B|.$$

Next we further refine the definition of exceptional vertices: an exceptional vertex $v \in A$ (similarly in B) is called *strongly exceptional* if it is connected to few vertices in the other set, more precisely if we have

$$deg(v, B) \leq \alpha^{1/3}|B|.$$

Denote the set of strongly exceptional vertices by $SE_A(\subset E_A)$ in A and by $SE_B(\subset E_B)$ in B. From (20) it is clear that

$$|SE_A| \leq 20\alpha|A| \text{ and } |SE_B| \leq 20\alpha|B|.$$

If we have a $u \in SE_A$ and a $v \in SE_B$, then we can exchange the two vertices and they will not be strongly exceptional anymore in their new sets. Note that after the exchanges for the vertices that were not strongly exceptional we still have $deg(v, B) \geq \alpha^{1/3}|B|/2$ which is sufficient for our purposes. Thus we may assume that one of the sets SE_A and SE_B is empty, since otherwise we may reduce the number of strongly exceptional vertices in both sets. Then we only have to eliminate the vertices in the non-empty set. For the remainder of this extremal case we will distinguish three subcases depending on the size of $\gamma = k/n$.

Case 1: $\gamma \leq \alpha^{1/3}$.

Without loss of generality we assume that $SE_B = \emptyset$ and $SE_A \neq \emptyset$ (the other case is similar). We may assume that we have no $u \in B$ with $deg(u, B) > \alpha^{1/3}|B|$, since otherwise we can exchange this vertex with a vertex $v \in SE_A$ and thus reducing the size of SE_A. Therefore any vertex $v \in B$ can be exchanged with any vertex $u \in SE_A$ without significantly changing the degree conditions. First for each $v \in SE_A$ we want to find a path P_v of length 2 such that the center of P_v is v, the other two vertices are in B and the paths are vertex disjoint for different v. Consider the graph $G' = G|_{SE_A \cup B}$, from (1) we have $\delta(G') \geq |SE_A|$ (in fact in this case the minimum degree is at least $k + |SE_A|$, but in the other case when $SE_B \neq \emptyset$ the minimum degree could be $|SE_B|$). From the size of SE_A and the fact that no vertex has high degree in B, we have

$\Delta(G') \leq 2\alpha^{1/3}|G'|$. Therefore by Lemma 5 there are at least SE_A vertex disjoint paths of length 2 in G'. Since every vertex in B can be exchanged with any vertex in SE_A, we can assume that all these paths have the center vertex in SE_A and the two end points in B.

Furthermore since the minimum degree in B is at least k and no vertex has degree more than $\alpha^{1/3}|B|$, by Lemma 5 we can find a matching of size k in B. From the fact that SE_A is very small, there exists a matching of size k that is vertex disjoint from all P_v selected above.

Then first we eliminate the paths of length 2 by embedding cycle parts into them. In this case for this purpose we may use cycles that either have even length at least 4, or odd length at least 7 (if we have to). Note that every vertex in B (in particular the endpoints of the paths P_v) have high degree to A, so any two vertices can be connected in one step using a vertex from A. Therefore, by a simple greedy procedure we can embed cycles in the bipartite graph between A and B, that use all P_v's and use exactly one edge inside B for each odd cycle. The remaining exceptional vertices (EA and EB) can also be used similarly, using the fact that their degree across is much larger than their number. Finally in the leftover almost-complete bipartite graph we may finish the embedding using Fact 16.

Case 2: $\alpha^{1/3} < \gamma \leq (\frac{1}{3} - \alpha^{1/3})$.

In this case we know that $SE_A = \emptyset$, as for each $v \in A$ we have

$$deg(v, B) \geq k > \alpha^{1/3}n > \alpha^{1/3}|B|.$$

Assume $SE_B \neq \emptyset$. First again we find a matching M of size k, such that at least one of the endpoints of each edge is non-exceptional. If we still have vertices left in SE_B (for simplicity let SE_B still denote the set of remaining vertices), then similarly as in Case 1, we find $|SE_B|$ vertex disjoint paths of length 2 such that the middle vertices are in SE_B and the other two vertices are from A. Then, as in Case 1, we first use the length-2 paths to embed a few cycles of length at least 4; the edges in M are used for the k odd cycles and we finish the embedding using Fact 16.

Case 3: $\gamma > (\frac{1}{3} - \alpha^{1/3})$.

In this case most of the cycles are triangles, indeed the number of vertices covered by cycles of length at least 4 is at most $8\alpha^{1/3}n$. Therefore in this case we will use triangles to eliminate the exceptional vertices. Assume first that the subgraph $G|_B$ is $\alpha^{1/4}$-non-extremal.

Let us consider an exceptional vertex $u \in A$ (note that we have $SE_A = \emptyset$) and its neighborhood in B of size at least $k \geq (\frac{1}{3} - \alpha^{1/3})n$. As $G|_B$ is non-extremal, we have edges inside this set and thus we can cover u with a triangle where the other two vertices come from B, as desired. For an exceptional vertex $v \in (B \backslash SE_B)$ again we can easily find a triangle where one of the other two vertices comes from A, the other from B. Finally let us take a vertex $v \in SE_B$. We may assume that we have no $u \in A$ with $deg(u, A) > \alpha^{1/3}|A|$, since otherwise we can exchange this vertex with v and thus reducing the size of SE_B. As in Case 1, since the minimum degree in the graph induced by $A \cup SE_B$ is at least $|SE_B|$ we can find $|SE_B|$ vertex disjoint edges going between A and SE_B. For each such edge e, since both of its endpoints have very high degree in B, we find a common neighbor in B for the endpoints to get a vertex disjoint triangle. Using Lemma 14 and the fact that $G|_B$ is $\alpha^{1/4}$-non-extremal, we can find a Hamiltonian cycle and thus a matching of size k' (the remaining number of odd cycles) in the leftover of B, and then Fact 16 finishes the embedding.

Finally let us assume that $G|_B$ satisfies Extremal Case 1 with $\alpha^{1/4}$ (the other extremal case is similar). So we have $V(G) = A \cup B \cup C$, $|A| = |B| = \frac{n-k}{2}$, $|C| = k$, so these three sets are roughly the same size with almost complete bipartite graphs between them. We will have two types of strongly exceptional vertices in each set; $v \in A$ is called *strongly exceptional to B* if it is connected to few vertices in B, more precisely if we have

$$deg(v, B) \leq \alpha^{1/3}|B|.$$

Denote the set of these vertices by SE_A^B. SE_A^C and the strongly exceptional sets in B and C are defined similarly. All strongly exceptional vertices can be handled similarly as above except for SE_A^C and SE_B^C. Indeed, for example to eliminate SE_A^B we can find $|SE_A^B|$ vertex disjoint edges going between SE_A^B and B and then we can close these edges to triangles by taking a third vertex from C.

Consider SE_A^C. Denote $k = n/3 - x$ where $x < \alpha^{1/3}n$, then we have $|A| = |B| = n/3 + x/2$ and $|C| = n/3 - x$. The goal is to embed the cycles in such a way that for each odd cycle C_i, one vertex is embedded to C and $(|C_i| - 1)/2$ vertices are embedded to A and B; for each even cycle C_i, $|C_i|/2$ vertices are embedded into A and B. Then we have at least $3x/2$ vertices in A (and symmetrically in B) for which we do not need any neighbors in C, the two neighbors along the corresponding embedded cycle will be in B (in A). Hence if $|SE_A^C| \leq 3x/2$, then we are done. Otherwise, denote $|SE_A^C| = \frac{3x}{2} + y$, where $y > 0$. Similarly as above by applying Lemma 5 we can find a matching of size at least y from SE_A^C to C (since in $G|_{SE_A^C \cup C}$ the minimum degree is at least y). These edges can be closed to triangles by taking a third vertex from B. The remaining at most $3x/2$ vertices in SE_A^C will be the vertices for which we need no neighbors in C, as above. This way we eliminate all the strongly exceptional vertices, and then by Fact 16 we can finish the embedding. This finishes Extremal Case 1.

4.2 Extremal Case 2

Here we have (1) and the following.

Extremal Case 2 (EC2) with parameter α: *There exists an $A \subset V(G)$ such that for $B = V(G) \setminus A$ we have*

- $\frac{n}{2} \geq |A| \geq \frac{n}{2} - \alpha n$, *and*
- $d(A, B) < \alpha$.

Thus roughly speaking, $G|_A$ and $G|_B$ are almost complete and the bipartite graph between A and B is sparse (note that k has to be small). By adding vertices to A we may achieve that $|A| = \lfloor n/2 \rfloor$ and $|B| = \lceil n/2 \rceil$. Furthermore, an easy computation shows that we still have $d(A, B) < 10\alpha$ (for simplicity we keep the notation A, B).

Again we define *exceptional* vertices $v \in A$ (and similarly for B), as

$$deg(v, A) \leq (1 - \sqrt{10\alpha})|A|.$$

Note that from the density condition $d(A, B) < 10\alpha$, the number of exceptional vertices in A is at most $\sqrt{10\alpha}|A|$ (and similarly for B). Let us denote the set of exceptional vertices by E_A in A and by E_B in B. Next again we further refine the definition of exceptional vertices: an exceptional vertex $v \in A$ (similarly in B) is called *strongly exceptional* if it is connected to few vertices in A, more precisely if we have

$$deg(v, A) \leq \alpha^{1/3}|A|.$$

Denote the set of strongly exceptional vertices by $SE_A (\subset E_A)$ in A and by $SE_B (\subset E_B)$ in B. If we have a $u \in SE_A$ and a $v \in SE_B$, then we can exchange the two vertices and they will not be strongly exceptional anymore in their new sets. Thus we may assume that one of the sets SE_A and SE_B is empty (say SE_B, the other case is similar). We first handle the vertices of SE_A.

We may assume that we have no $u \in B$ with $deg(u, A) > \alpha^{1/3}|A|$, since otherwise we can exchange this vertex with a vertex $v \in SE_A$ and thus reducing the size of SE_A. We remove the vertices in SE_A from A and add them to B, and denote the resulting sets by A' and B'. It is easy to see using (1) that in $G|_{A'}$ apart from at most $10\sqrt{\alpha}|A'|$ exceptional vertices all the degrees are at least $(1-10\sqrt{\alpha})|A'|$, and the degrees of the exceptional vertices are at least $\alpha^{1/3}|A'|/2$. In $G|_{B'}$ we have an even stronger degree condition; all the degrees are at least $(1-2\alpha^{1/3})|B'|$.

Suppose we have our cycles listed in increasing order of size, $C_1, C_2, \dots C_r$. We assign cycles to A' until we have no room left and denote by C_m the last cycle, i.e. by adding this cycle we have at least $|A'|$ vertices assigned to $|A'|$, but without this cycle we have fewer than $|A'|$ assigned vertices. We refer to C_m as the *middle cycle* and let $n_m = |C_m|$. Denote

$$n_m^1 = |A'| - \sum_{i=1}^{m-1} |C_i| \text{ and } n_m^2 = n_m - n_m^1.$$

Note that part of C_m has to be embedded into A' (n_m^1 vertices) while the other part (n_m^2 vertices) into B'.

We may assume that $n_m^2 > 0$, since otherwise we are done. If $n_m^1, n_m^2 \geq 3$, then it is easy to embed the middle cycle C_m. Indeed we can find two bridge edges (u_i, v_i) with $u_i \in (A \setminus E_A)$, $v_i \in B$ for $i = 1, 2$ since from (1) we have $deg(u, B) \geq 1$ for each $u \in A$ and $deg(v, A) \leq \alpha^{1/3}|A|$ for each $v \in B$. Then we can connect u_1 and u_2 with a path of length $n_m^1 - 1$ (counting edges) in A and v_1 and v_2 with a path of length $n_m^2 - 1$ in B. Actually this argument works for $n_m^1 = 2, n_m^2 \geq 3$ as well, since we can have two bridge edges where (u_1, u_2) is also an edge in $G|_A$. For $n_m^2 = 2$ note that if we can find two vertex disjoint paths of length 2 such that the center vertices are in B and the endpoints are in $A \setminus E_A$, then we move the center vertices to A' and now we have a perfect assignment. These length 2 paths can be used as part of some cycles and hence we are done. Thus we may assume that we have no two such paths. However, this fact and the degree conditions imply that we can find two bridges where (u_1, u_2) and (v_1, v_2) are *both* edges taking care of all cases $n_m^1, n_m^2 \geq 2$.

Finally let us assume that $n_m^1 = 1, n_m^2 \geq 2$ (the other case is symmetric). If we can find a path of length 2 with its center vertex in A' and endpoints in B', and then we can move the center vertex to B' to have a perfect assignment. Thus we may assume that $k = 0$ (so n is even, all cycles are even and thus $n_m^2 > 2$) and $|A| = |A'| = |B| = |B'|$ and we have no such path of length 2. Then $G|_A$ and $G|_B$ are both complete graphs with a perfect matching M between them and all of our cycles are even. Let $n_m = 2s, s \geq 2$, we will find a cycle $C_i : i < m, |C_i| = 2p, 2 \leq p < s$ or a cycle $C_j : j > m, |C_j| = 2q, q > s$. In case we have one such a cycle (say C_i), we embed p vertices of C_i in A as a path while the other p vertices in B as a path, and then joining the endpoints using two edges from M. Now in the remaining graph and cycle system we have $n_m^1 = p + 1 \geq 2$, while $n_m^2 = 2s - 1 - p \geq 2$, so C_m can be easily embedded using two edges from M. Note that we can always find either C_j or C_i, to see this, assume there are no such cycle, then all cycles are of length $2s$, hence $n \equiv 0 \pmod{2s}$ while $|A| = n/2 \equiv 1 \pmod{2s}$, which is a contradiction for $s > 1$.

It is easy to see that the other cycles apart from C_m (and possibly C_i or C_j in the last case) assigned to A' (and B') can be embedded in $G|_{A'}$ (and in $G|_{B'}$) by eliminating the few exceptional vertices first and then applying Fact 16. This finishes EC2.

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