Dynamic Programming

- The Knapsack Problem
- Dynamic Programming Formulation
- Implementation
- Fractional Knapsack and Subset Sum Problem

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Knapsack Problem: Dynamic Programming

Input: A set $\mathcal{U}$ of objects $\{a_1, \ldots, a_n\}$ with

- integral weights $\{w_1, \ldots, w_n\}$ and
- positive values $\{v_1, \ldots, v_n\}$ and
- a positive integer C (capacity)

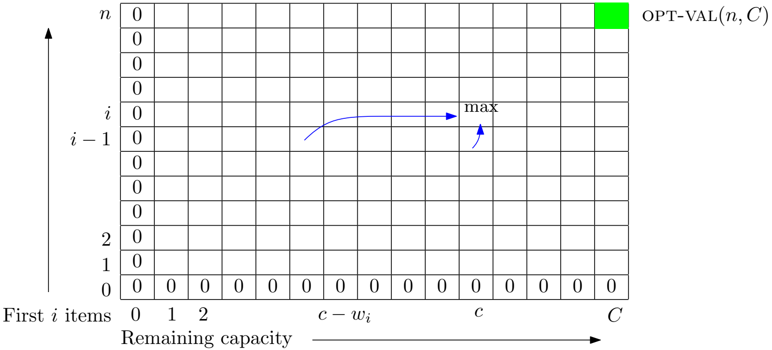
Output: A subset $S \subset \mathcal{U}$ with total weight $\leq C$ and maximum total value

- Fix an order on objects $a_1, \ldots, a_n$
- $\text{OPT-SET}(k, c)$ is the max value feasible subset of $\mathcal{U}[1 \ldots k]$ and c
- $\text{OPT-VAL}(k, c)$ is the total value of $\text{OPT-SET}(k, c)$
- Out goal is to find $\text{OPT-SET}(n, C)$ (and $\text{OPT-VAL}(n, C)$)

Knapsack Problem: Dynamic Programming

$$\textsc{opt-val}(k, c) = \max \begin{cases} 0 & \text{if } k = 0 \\ 0 & \text{if } c = 0 \\ \textsc{opt-val}(k-1, c-w_k) & \text{if } a_k \in \textsc{opt-set}(k, c) \\ \textsc{opt-val}(k-1, c) & \text{if } a_k \notin \textsc{opt-set}(k, c) \end{cases}$$

- This is a two variable recurrence. Need a 2-dimensional memo



Knapsack Problem: Dynamic Programming

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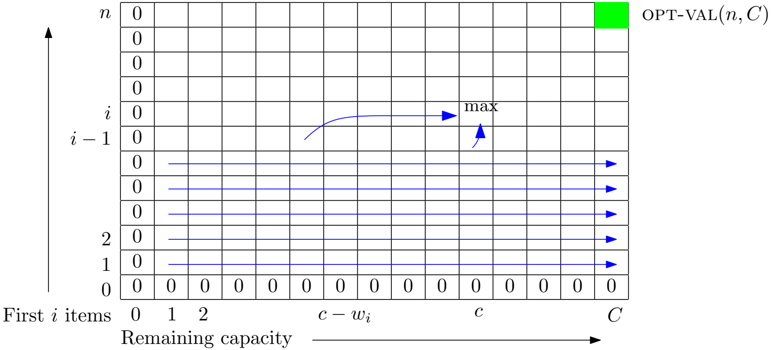
- Fill in the memo solutions table bottom to top, left to right



Knapsack Problem: Dynamic Programming

Algorithm Knapsack with memoization, n, C

for $i = 0$ to n **do** ▷ Initially $\text{OPT-SET}[i][c]$'s are unknown
  for $c = 0$ to C **do**
    $\text{OPT}[i][c] \leftarrow -\infty$

for $c = 0$ to C **do**
  $\text{OPT}[0][c] \leftarrow 0$ ▷ when $i = 0 \implies U = \emptyset$, then $\text{OPT}[0][\cdot] = 0$

for $i = 0$ to n **do**
  $\text{OPT}[i][0] \leftarrow 0$ ▷ when $c = 0 \implies$ no capacity, then $\text{OPT}[\cdot][0] = 0$

for $i = 1$ to n **do**
  for $c = 0$ to C **do**
    if $\text{OPT}[i-1][c-w_i] + v_i \geq \text{OPT}[i-1][c]$ and $c \geq w_i$ **then**
      $\text{OPT}[i][c] \leftarrow \text{OPT}[i-1][c-w_i] + v_i$
    else
      $\text{OPT}[i][c] \leftarrow \text{OPT}[i-1][c]\}$

return $\text{OPT}[n][C]$

Knapsack Problem: Dynamic Programming

$$\textsc{opt-val}(k,c) = \max \begin{cases} 0 & \text{if } k = 0 \\ 0 & \text{if } c = 0 \\ \textsc{opt-val}(k-1, c-w_k) & \text{if } a_k \in \textsc{opt-set}(k,c) \\ \textsc{opt-val}(k-1, c) & \text{if } a_k \notin \textsc{opt-set}(k,c) \end{cases}$$

- $a_i \in \textsc{opt-set}(i,c)$ iff $\textsc{opt-val}[i-1][c-w_i] + v_i \geq \textsc{opt-val}[i-1][c]$
- Backtrack from $\textsc{opt-val}(n, C)$ to see whether or not a_i is included

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function $\textsc{find-set}(i,c)$
  if $i = 0$ or $c = 0$ **then**
    return $\emptyset$
  else
    if $OptVal[i-1][c-w_i] + v_i \geq OptVal[i-1][c]$ **then**
      return $a_i \cup \textsc{find-set}(i-1, c-w_i)$
    else
      return $\textsc{find-set}(i-1, c)$

Knapsack Problem: Dynamic Programming

function FIND-SET(i,c)
  if $i = 0$ or $c = 0$ **then**
    return $\emptyset$
  else
    if $OptVal[i-1][c-w_i] + v_i \geq OptVal[i-1][c]$ **then**
      return $a_i \cup$ FIND-SET$(i-1, c-w_i)$
    else
      return FIND-SET$(i-1, c)$

5	0	3	3	4	4	8	11	11	12	12	12	13	13	13	16	16
4	0	3	3	4	4	8	11	11	12	12	12	13	13	13	13	13
3	0	3	3	3	3	8	11	11	11	11	11	13	13	13	13	13
2	0	3	3	3	3	3	3	5	5	5	5	5	5	5	5	5
1	0	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

ID	weight	value
a_1	1	3
a_2	6	2
a_3	5	8
a_4	2	1
a_5	8	5

$C = 15$

Knapsack Problem: Dynamic Programming

Runtime:

- Each entry is filled in $O(1)$ if the two required entries are already filled
- FIND-SET(n, C) takes $O(n)$ time
- Total runtime is $O(nC)$
- pseudo-polynomial time
- C is the input, not size of input
- C can be expressed in $\log C$ bits
- So it is exponential in size of one input parameter
- Note we required C to be integer, as memo is indexed by it