Single Source Shortest Path

- Weighted Graphs and Shortest Paths
- Dijkstra Algorithm
- Proof of Correctness
- Runtime
  - Basic Implementation
  - Vertex-Centric Implementation
  - Heap Based Implementation

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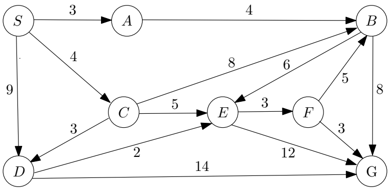
Weighted Graph

Weighted Graphs (digraphs)

- V : Set of vertices
- E : Set of edges (directed edges)
- w : cost/weight function: $w : E \to \mathbb{R}$
- weights could be lengths, airfare, toll, energy
- Denoted by $G = (V, E, w)$

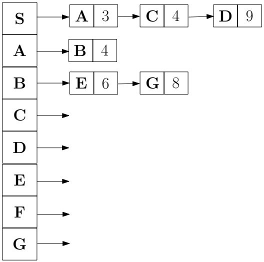
Weighted Graph Representation



Weighted Adjacency Matrix

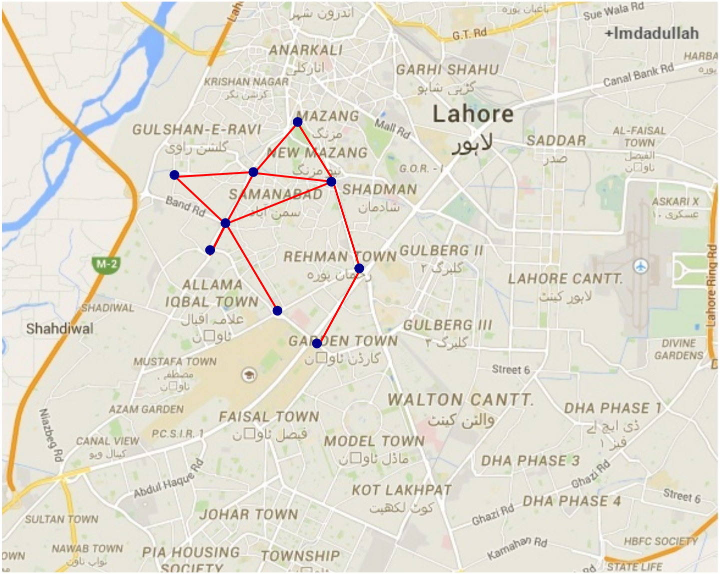
	S	A	B	C	D	E	F	G
S	0	3	0	4	9	0	0	0
A	0	0	4	0	0	0	0	0
B	0	0	0	0	0	6	0	8
C	⋮							
D								
E								
F								
G								

Weighted Adjacency Lists

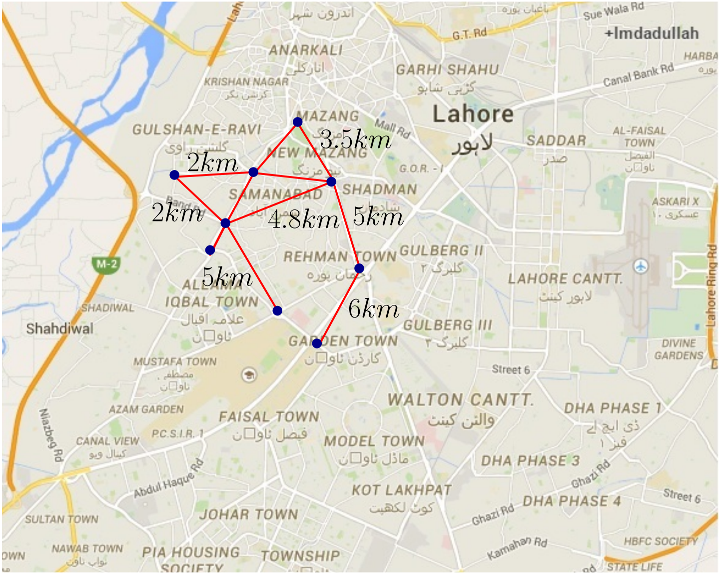


Weighted Graph

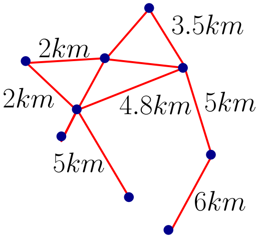
Weighted Graph



Weighted Graph



Weighted Graph



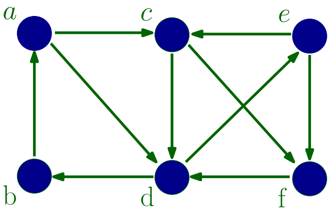
Paths in Graphs

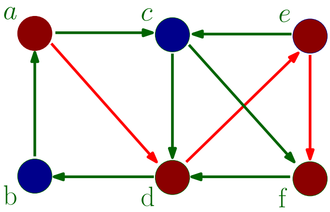
A path in a digraph is a sequence of vertices with no repetition

$$v_1, v_2, \ldots, v_k$$

such that $(v_i, v_{i+1}) \in E$ for $1 \leq i \leq k-1$

Length of the path is the number of edges in it



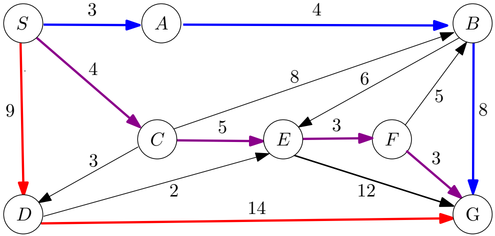


a, d, e, f

Weight of Paths

Weight or length of a path $p = v_0, v_1, \ldots, v_k$ in weighted graphs is sum of the weights of its edges

$$C(p) \ = \ \sum_{i=1}^{k} w(v_{i-1}, v_i)$$



Three $S - G$ paths

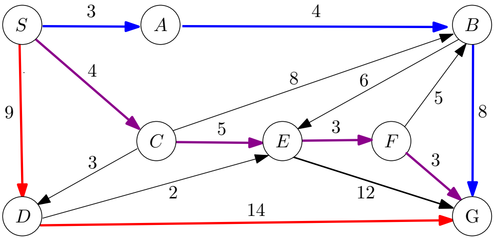
$\mathbf{C(p_1) = 3 + 4 + 8}$

$\mathbf{C(p_2) = 4 + 5 + 3 + 3}$

$\mathbf{C(p_3) = 9 + 14}$

Unweighted graphs are weighted graphs with unit edge weights

Shortest Paths



Three $S - G$ paths

$\mathbf{C(p_1) = 3 + 4 + 8}$

$\mathbf{C(p_2) = 4 + 5 + 3 + 3}$

$\mathbf{C(p_3) = 9 + 14}$

Shortest path from s to t is a path of smallest weight

Distance from s to t, $\mathbf{d(s, t)}$: weight of the shortest $s - t$ path

There can be multiple shortest paths

Shortest Path Problems

1. Shortest $s - t$ path:

   Given $G = (V, E, w)$ and $s, t \in V$, find a shortest path from s to t

   - For undirected graph, it will be a path between s and t
   - Unweighted graphs are weighted graphs with all edge weights $= 1$
   - Shortest path is not unique, any path with minimum weight will work

2. Single source shortest paths (SSSP):

   Given $G = (V, E, w)$ and $s \in V$, find shortest paths from s to all $t \in V$

   - Problems of undirected and unweighted graphs are covered as above
   - It includes the first problem

We focus on SSSP