Asymptotic Analysis

- Runtime Analysis and Big Oh - $O(\cdot)$
- Complexity Classes and Curse of Exponential Time
- $\Omega(\cdot)$, $\Theta(\cdot)$, $o(\cdot)$, $\omega(\cdot)$ - Relational properties

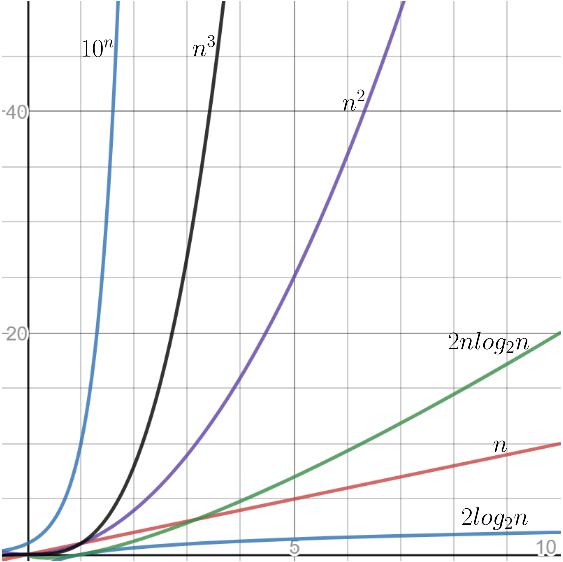
Imdad ullah Khan

Asymptotic-Complexity Classes

Class Name	**Class Symbol**	**Example**
Constant	$O(1)$	Comparison of two integers
Logarithmic	$O(log(n))$	Binary Search, Exponentiation
Linear	$O(n)$	Linear Search
Log-Linear	$On(log(n))$	Merge Sort
Quadratic	$O(n^2)$	Integer multiplications
Cubic	$O(n^3)$	Matrix multiplication
Polynomial	$O(n^a)$, $a \in \mathbb{R}$	
Exponential	$O(a^n)$, $a \in \mathbb{R}$	Print all subsets
Factorial	$O(n!)$	Print all permutations

$$n! \gg 2^n \gg n^3 \gg n^2 \gg nlogn \gg n \gg logn \gg 1$$

Growth Rates of Functions



Find F_n: The curse of Exponential time

Fibonacci Sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

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$$F_n = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ F_{n-1} + F_{n-2} & \text{if } n > 2 \end{cases}$$

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Implementation of the recursive definition of F_n

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- Is it correct?
- How much time it takes to compute F_n?
- Can we do better?

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$T(n) \geq 2^{.69n}$ ▷ **exponential** in n (prove by induction)

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Another perspective to see growth of exponential time

- Runtime of FIB1(n) is $\geq 2^{0.694n} \approx (1.6)^n$
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How can we improve it?

Exponential vs Polynomial Growth rates

Sizes of problems that can be solved within 10^{12} operations on today's computer and next years computer with double speed

Complexity	Increase	Problem Size (today)	Problem Size (next year)
n	$n \to 2n$	10^{12}	2×10^{12}
n^2	$n \to \sqrt{2}n$	10^6	1.4×10^6
n^3	$n \to \sqrt[3]{2}n$	10^4	1.25×10^4
$2^{n/10}$	$n \to n + 10$	400	410
2^n	$n \to n + 1$	40	41