Approximation Algorithms

- Approximation Algorithms for Optimization Problems: Types
- Absolute Approximation Algorithms
- Inapproximability by Absolute Approximate Algorithms
- Relative Approximation Algorithm
- InApproximability by Relative Approximate Algorithms
- Polynomial Time Approximation Schemes
- Fully Polynomial Time Approximation Schemes

Imdad Ullah Khan

Relative Approximation Algorithms

For algorithm A for an optimization problem with value function $f(> 0)$,

Approximation Ratio of A is $max\left\{ f\left(A(I)\right)/f\left(\mathbf{opt}(I)\right), f\left(\mathbf{opt}(I)\right)/f\left(A(I)\right) \right\}$

$\triangleright$ For minimization problem it is $f\left(A(I)\right)/f\left(\text{OPT}(I)\right)$

$\triangleright$ For maximization problem it is $f\left(\text{OPT}(I)\right)/f\left(A(I)\right)$

A is called an α-**approximate** algorithm

if for any instance I of size n, A achieves an approximation ratio α

$\triangleright$ For minimization problems this means $f\left(A(I)\right) \leq \alpha \cdot f\left(\text{OPT}(I)\right)$

$\triangleright$ For maximization problems this means $f\left(A(I)\right) \geq 1/\alpha \cdot f\left(\text{OPT}(I)\right)$

When α does not depend on n, A is called constant factor (relative) approximation algorithm

- Given U of n elements
- A collection $\mathcal{S}$ of m subsets of U, $\quad S_1, S_2, \ldots, S_m$
- A Set Cover is a subcollection $I \subset \{1, 2, \ldots, m\}$ with $\bigcup_{i \in I} S_i = U$

$U = \{1, 2, 3, 4, 5, 6\}$

Sets: $\{1, 2, 3\}, \{3, 4\}, \{1, 3, 4, 5\}, \{2, 4, 6\}, \{1, 3, 5, 6\}, \{1, 2, 4, 5, 6\}$

Cover $\{1, 2, 3\}, \quad \{1, 3, 4, 5\}, \{2, 4, 6\}$

Cover $\{1, 2, 3\}, \quad \{1, 2, 4, 5, 6\}$

Cover $\quad \{1, 3, 4, 5\}, \quad \{1, 2, 4, 5, 6\}$

The first cover has size 3, the latter two have size 2 each

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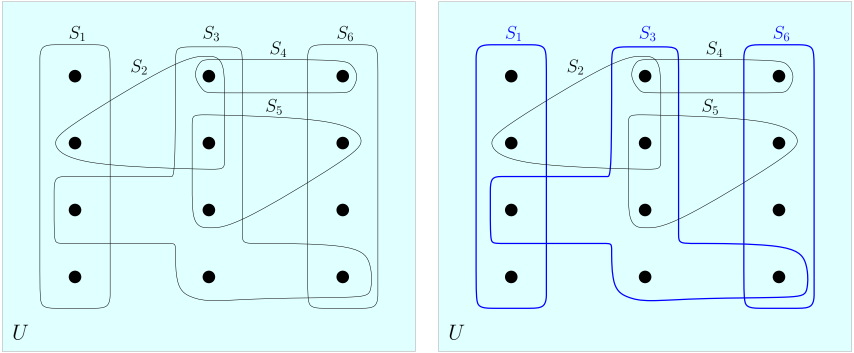

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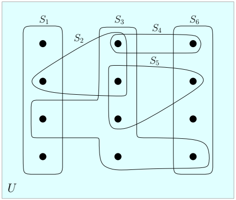

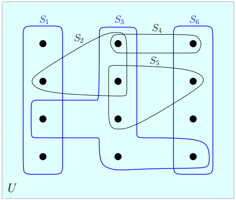

The MIN-SET-COVER$(U, \mathcal{S})$ problem: Find a cover of minimum size?

In the more general version, each set in $\mathcal{S}$ has a weight/cost and the goal is to find a cover with minimum total weight

SET-COVER: Greedy Approximation Algorithm

While there is an uncovered element in U, choose a set S_i from $\mathcal{S}$ that covers the most number of (yet) uncovered elements

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Algorithm GREEDY-SET-COVER$(U, \mathcal{S})$

$X \leftarrow U$ ▷ Yet uncovered elements
$C \leftarrow \emptyset$
while $X \neq \emptyset$ **do**
  Select an $S_i \in \mathcal{S}$ that maximizes $|S_i \cap X|$ ▷ Covers most elements
  $C \leftarrow C \cup S_i$
  $X \leftarrow X \setminus S_i$
return C

- $U = \{1,2,3,4,5\}$, $\mathcal{S} = \Big\{\{1,2\},\{1\},\{1,4\},\{4\},\{1,2,3,5\},\{4,5\}\Big\}$
- First pick $\{1,2,3,5\}$ as it covers 4 elements
- Next pick $\{1,4\}$, $\{4\}$ or $\{4,5\}$ to cover all elements of U

SET-COVER: Greedy Approximation Algorithm

Algorithm GREEDY-SET-COVER$(U, \mathcal{S})$

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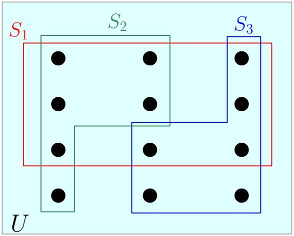
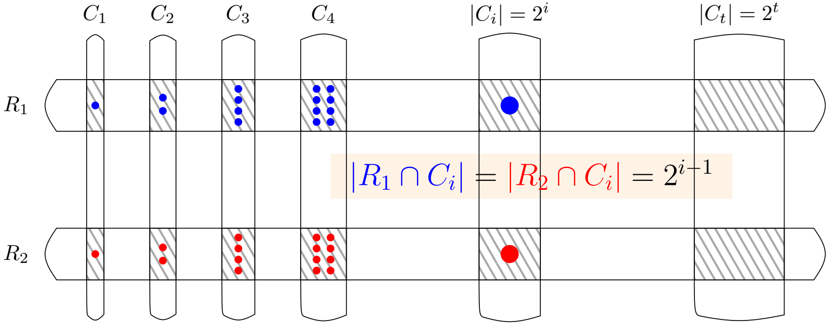


- The algorithm will select S_1, S_2, and S_3. While optimal is S_2 and S_3

SET-COVER: Greedy Approximation Algorithm

- Let k be the size of an optimal set cover
- By pigeon-hole principle some set $S \in \mathcal{S}$ covers at least ${}^{n}/_{k}$ elements
- Let n_i be the number of uncovered elements after ith iteration $\triangleright |X|$
- There is a set $S \notin C$ covering at least ${}^{n_i}/_{k}$ elements
  - Actually there will be a set covering at least ${}^{n_i}/_{k-i}$ elements
- We get $n_i \leq (1 - {}^{1}/_{k})n_{i-1} \leq (1 - {}^{1}/_{k})^2 n_{i-2} \leq \ldots \leq (1 - {}^{1}/_{k})^i n$
- The algorithm stops after t iterations when $n_t \leq (1 - {}^{1}/_{k})^t n < 1$
- This happens when $t = k \ln n$
- Approximation ratio of this algorithm is $O(\log n)$

SET-COVER: Greedy Approximation Algorithm

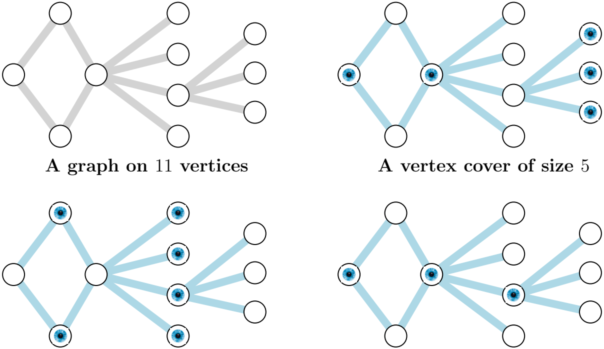


$|R_1| = |R_2| = \sum_{i=1}^{t} 2^{i-1} = 2^t - 1$

$n = |U| = \left| \bigcup_{i=1}^{t} C_i \right| = \sum_{i=1}^{t} 2^i = 2^{t+1} - 2$

- GREEDY-SET-COVER selects $C_t, C_{t-1}, \ldots, C_1$
- The optimal solution is R_1 and R_2
- On this example, the algorithm approximation factor is $O(\log n)$
- Unless $\mathrm{P} = \mathrm{NP}$, this is the best approximation guarantee

An vertex cover in a graph is subset C of vertices such that each edge has at least one endpoint in C



A graph on 11 vertices

A vertex cover of size 5

A vertex cover of size 6

A vertex cover of size 3

The MIN-VERTEX-COVER(G) problem: Find a min vertex cover in G?

VERTEX-COVER: Greedy Algorithm

- The greedy idea is to keep adding vertices that cover maximum edges

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Algorithm GREEDY-VERTEX-COVER(G)

$C \leftarrow \emptyset$

while $E(G) \neq \emptyset$ **do**

  Select v that has maximum degree

  $C \leftarrow C \cup \{v\}$

  $G \leftarrow G - v$

return C

- Essentially graph version of GREEDY-SET-COVER$(U, \mathcal{S})$ algorithm
- Clearly returns a vertex cover and is $O(\log n)$-approximate algorithm

- The greedy idea is to keep adding vertices that cover maximum edges

Algorithm GREEDY-VERTEX-COVER(G)

$C \leftarrow$ *emptyset*
while $E(G) \neq \emptyset$ **do**
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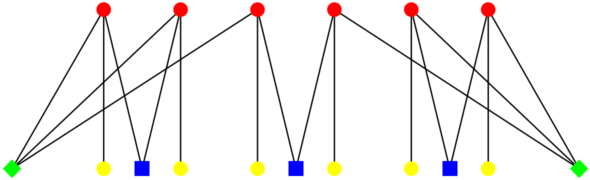
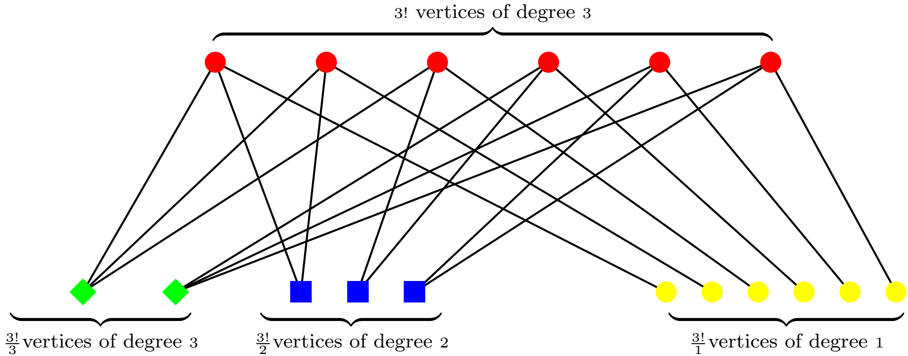


- Depending on tie-breaking, the algorithm could select the
- the 2 green vertices, 3 blue vertices, then 6 red vertices $\triangleright\ |C| = 11$
- While minimum vertex cover is of size 6 (red vertices)

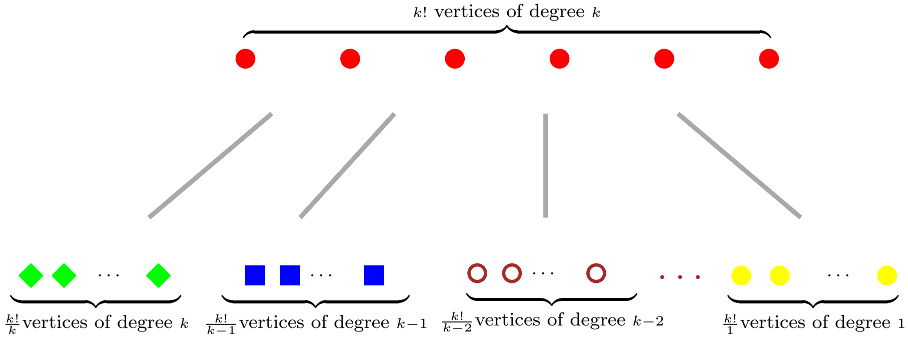
- The greedy idea is to keep adding vertices that cover maximum edges
- Another view of the above example



OPT-Cover : Top Vertices: 3!

Greedy Cover: Bottom Vertices: $3!(\frac{1}{3} + \frac{1}{2} + \frac{1}{1})$

- The greedy idea is to keep adding vertices that cover maximum edges
- A tight example for the greedy algorithm



OPT-Cover : Top Vertices: $k!$

Greedy Cover: Bottom Vertices: $k!(\frac{1}{k} + \frac{1}{k-1} + \ldots + \frac{1}{1}) = k! \log k$

- VERTEX-COVER is a special case, we exploit it's special structure
- Note: for every edge (x, y), x or y or both have to be in optimal cover

Algorithm VERTEX-COVER(G)

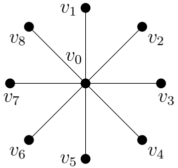
$C \leftarrow \emptyset$
while $E \neq \emptyset$ **do**
  pick any $\{u, v\} \in E$, select arbitrarily u or v (call it s)
  $C \leftarrow C \cup \{s\}$
  Remove all edges incident to s
return C

VERTEX-COVER(G) clearly produces a cover

Output could be very arbitrarily bad

Optimal cover is $\{v_0\}$

Output could be all other vertices



- Note: for every edge (x, y), x or y or both have to be in optimal cover
- A better approximation uses the seemingly wasteful idea

Algorithm VERTEX-COVER(G)

$C \leftarrow \emptyset$
while $E \neq \emptyset$ **do**
  pick any $\{u, v\} \in E$
  $C \leftarrow C \cup \{u, v\}$
  Remove all edges incident to either u or v
return C

VERTEX-COVER(G) clearly produces a cover, how good is it?

Algorithm vertex-cover(G)

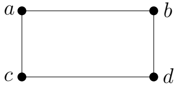
$C \leftarrow \emptyset$
while $E \neq \emptyset$ **do**
  pick any $\{u, v\} \in E$
  $C \leftarrow C \cup \{u, v\}$
  Remove all edges incident to either u or v
return C

VERTEX-COVER(G) clearly produces a cover

How good is it?

VERTEX-COVER(G) is 2-approximate

- For each edge $e = (u, v)$, OPT must include either u or v
- At worst VERTEX-COVER(G) picks both u and v ▷ $f(C) \leq 2f(\text{OPT})$



- An optimal cover is $\{a, d\}$
- We may choose $\{a, b, c, d\}$

- Best known guarantee for vertex cover is $2 - O(\log\log n/ \log n)$
- The best known lower bound is $4/3$ ▷ Open problem: close the gap

Scheduling on Identical Parallel Machines

- This is a general problem of load balancing
- An instance of the scheduling problem consists of
  - P : Set of n jobs (processes) $\{p_1, p_2, \cdots, p_n\}$
  - Each job p_i has a processing time t_i
  - M : Set of k identical machines $\{m_1, m_2, \cdots, m_k\}$
- A schedule, $S : P \to M$ is an assignment of jobs to machines
- Let $A(j)$ be set of jobs assigned to m_j (preimages of m_j)
- Load L_j of machine m_j is the total time of processes assigned to it

$$L_j = \sum_{p_i \in A(j)} t_i$$

- MAKESPAN of a schedule is the maximum load of any machine
- $\text{MAKESPAN}(S) = \max_{m_j} L_j$

Instance: $[P, M]$

- P : Set of n jobs $\{p_1, p_2, \cdots, p_n\}$ each with time t_i
- M : Set of k identical machines $\{m_1, m_2, \cdots, m_k\}$

- A schedule, $S : P \to M$ is an assignment of jobs to machines
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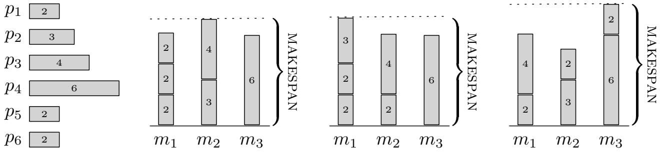

MIN-MAKESPAN(P, M) problem: Find a schedule S with min MAKESPAN(S)

The decision version MIN-MAKESPAN(P, M, t) is NP-COMPLETE

- List scheduling [Graham (1966)] is a simple greedy algorithm
- Go through jobs one by one in some fixed order
- Assign p_i to a machine that currently has the lowest load

Algorithm List Scheduling Algorithm

for $j = 1 : k$ **do**
  $A_j \leftarrow \emptyset$
  $L_j \leftarrow 0$
for $i = 1 \rightarrow n$ **do**
  m_j : machine with minimum load at this time: $m_j = \arg\min_j L_j$
  $A_j \leftarrow A_j \cup p_i$
  $L_j \leftarrow L_j + t_i$

- The first approximation algorithm (with proper worst case analysis)

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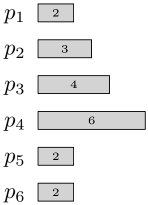

m_1 m_2 m_3

order $2, 3, 4, 6, 2, 2$

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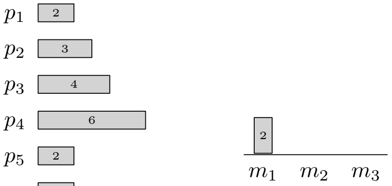

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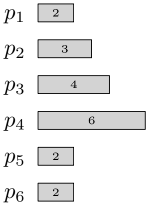

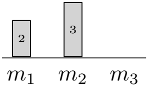

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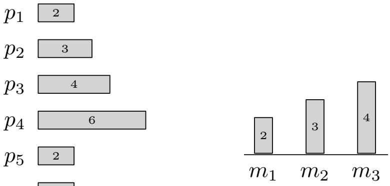

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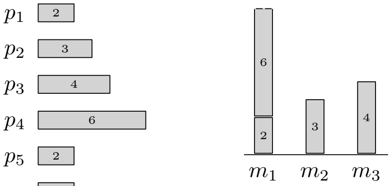

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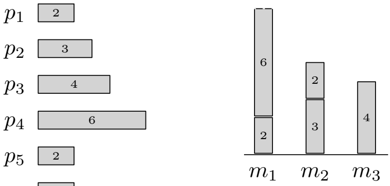

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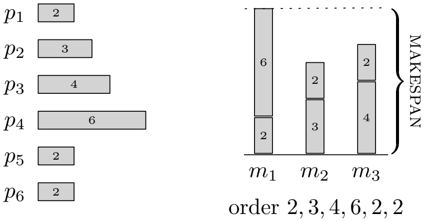

- If the order of jobs is $2, 3, 4, 6, 2, 2$ $\triangleright$ $L_1 = 8$
-
-

Algorithm List Scheduling Algorithm

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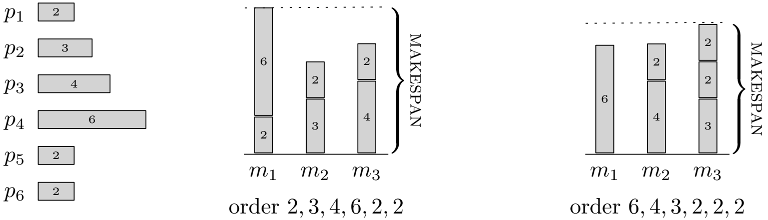


- If the order of jobs is $2, 3, 4, 6, 2, 2$ $\triangleright$ $L_1 = 8$
- If the order of jobs is $6, 4, 3, 2, 2, 2$ $\triangleright$ $L_3 = 7$ (optimal)
- Notice that order is very critical

Let $I = [P, M]$ be an instance

We get the following lower bounds

$$\text{OPT}(I) \;\geq\; \max_{p_i \in P} t_i \;=\; t_{max}$$

$\triangleright$ $\because$ the machine getting the longest process will have load at least t_{max}

$$\text{OPT}(I) \;\geq\; \frac{1}{k} \sum_i t_i$$

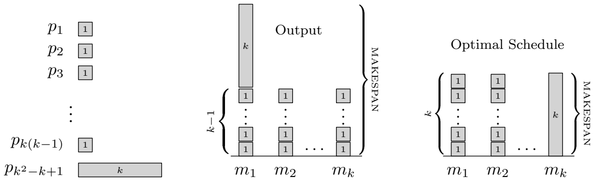
$\triangleright$ By PHP one of the k machines must do at least $\frac{1}{k} \sum_i t_i$ total work

$$\text{OPT}(I) \geq \max_{p_i \in P} t_i = t_{max} \qquad \text{and} \qquad \text{OPT}(I) \geq \frac{1}{k}\sum_i t_i$$

- WLOG say m_1 has max load and let p_i be the last job placed at m_1
- At the time p_i (iteration i) was assigned to m_1, load of m_1 was lowest
- Let L_1' be the load of m_1 at the time of assigning p_i
- p_i is the last job placed at $m_1 \implies L_1' = L_1 - t_i$
- m_1 was least loaded at time i, so for all other machines $L_j \geq L_1 - t_i$
- $\sum_{m_j \in M} L_j = \sum_{p_i \in P} t_i \geq k(L_1 - t_i) + t_i$
- $\text{OPT}(I) \geq \frac{1}{k}\sum_{p_i \in P} t_i \geq \frac{1}{k}(k(L_1 - t_i) + t_i) = L_1 - (1 - 1/k)\, t_i$
- $\text{OPT}(I) \geq L_1 - (1 - 1/k)\,\text{OPT}(I)$ ▷ First Lower bound
- $\text{MAKESPAN}(A(I)) = L_1 \leq (2 - 1/k)\,\text{OPT}(I)$

The LIST SCHEDULING ALGORITHM is $(2 - 1/k)$-approximate

- This analysis is tight
- $k(k-1)+1$ jobs. Time of first $k(k-1)$ jobs is 1. Time of last is k
  - $k(k-1)$ jobs of time 1 scheduled on each machine in round-robin fashion
  - Then the last job will be scheduled on any one machine



- OPT: First $k(k-1)$ jobs uniformly on $k-1$ machines, last job to M_k
- The achieved approximation factor is $\frac{2k-1}{k} = 2 - 1/k$

- The example show that we should not delay assigning long processes
- Graham (1969): Longest Processing Time First (LPT rule)
- Go through jobs one by one in ~~some fixed~~ decreasing order
- Assign p_i to a machine that currently has the lowest load

Algorithm List Scheduling Algorithm with LPT (P, M)

$\textsc{sort}(P)$ so that $t_1 \geq t_2 \ldots \geq t_n$
for $j = 1 : k$ **do**
    $A_j \leftarrow \emptyset$
    $L_j \leftarrow 0$
for $i = 1 \rightarrow n$ **do**
    m_j : machine with minimum load at this time: $m_j = \arg\min_j L_j$
    $A_j \leftarrow A_j \cup p_i$
    $L_j \leftarrow L_j + t_i$

- **[LB-1]** $\text{OPT}(I) \geq \max_{p_i \in P} t_i = t_{max}$
- **[LB-2]** $\text{OPT}(I) \geq \frac{1}{k}\sum_i t_i$
- If $n \leq k$, then list scheduling gives optimal solution

  Assume $n > k$, then with LPT, a tighter lower bound is:

- **[LB-3]** $\text{OPT}(I) \geq 2t_{k+1}$
- Since $t_1 \geq t_{k-1} \geq t_k \geq t_{k+1}$
- Some machine must get at least two jobs among the first $k+1$ jobs, its load will be $\geq 2t_{k+1}$

- **[LB-1]** $\text{OPT}(I) \geq \max_{p_i \in P} t_i = t_{max}$
- **[LB-2]** $\text{OPT}(I) \geq \frac{1}{k}\sum_i t_i$
- **[LB-3]** $\text{OPT}(I) \geq 2t_{k+1}$ ▷ Assuming $n > k$
- WLOG say m_1 has max load and let p_i be the last job placed at m_1
- At the time p_i (iteration i) was assigned to m_1, load of m_1 was lowest
- Let L'_1 be the load of m_1 at time i, $L'_1 = L_1 - t_i$
- For all j, $L_j \geq L - t_i$, $\therefore \sum_{m_j \in M} L_j = \sum_{p_i \in P} t_i \geq k(L_1 - t_i) + t_i$
- $\text{OPT}(I) \geq {}^1\!/_k \sum_{p_i \in P} t_i \geq {}^1\!/_k (k(L_1 - t_i) + t_i) = L_1 - (1 - {}^1\!/_k)\, t_i$
- $\text{OPT}(I) \geq L_1 - (1 - {}^1\!/_k)\, {}^1\!/_2\, \text{OPT}(I)$ ▷ **[LB-3]**
- $\text{MAKESPAN}(A(I)) = L_1 \leq ({}^3\!/_2 - {}^1\!/_{2k})\, \text{OPT}(I)$

The List Scheduling Algorithm with LPT is $(3/2 - 1/2k)$-approximate

- This analysis is not tight - A more sophisticated analysis yields

The List Scheduling Algorithm with LPT is $(4/3 - 1/3k)$-approximate

- This analysis is tight, consider $2k+1$ jobs
- 3 of duration k and 2 each of $k+i$, $1 \leq i \leq k-1$
- The algorithm gives all but one machine 2 jobs with total load $3m-1$
- The remaining machine gets 3 jobs and load $4m-1$
- OPT: 3 length-k jobs on a machine and remaining loads are $3k$
- The achieved approximation factor is $^{4k-1}/_{3k} = {}^{4}/_{3} - {}^{1}/_{3k}$