Randomized Computation

- Deterministic and (Las Vegas & Monte Carlo) Randomized Algorithms
- Probability Review
- Probabilistic Analysis of deterministic QUICK-SORT Algorithm
- RANDOMIZED-SELECT and RANDOMIZED-QUICK-SORT
- Max-Cut
- Min-Cut
- MAX-3-SAT and Derandomization
- Closest Pair
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The MAX-3-SAT Problem

- Given n Boolean variables $x_1, \ldots, x_n$
- Each can take a value of $0/1$ (true/false)
- A literal is a variable appearing in some formula as x_i or $\bar{x}_i$
- A clause of size 3 is an OR of three literals
- A 3-CNF formula is AND of one or more clauses of size ≤ 3
- A formula is satisfiable if there is an assignment of $0/1$ values to the variables such that the formula evaluates to 1 (or true)

3-SAT(f) problem: Is there a satisfying assignment for 3-CNF formula f?

MAX-3-SAT(f) problem: Find an assignment for 3-CNF formula f that satisfies the maximum number of clauses

MAX-3-SAT(f) problem: Find an assignment for 3-CNF formula f that satisfies the maximum number of clauses

- The problem is NP-HARD
- Brute Force: Try all 2^n possible assignments in $\mathcal{O}(m2^n)$

$\triangleright$ m is the number of clauses

MAX-3-SAT(f) problem: Find an assignment for 3-CNF formula f that satisfies the maximum number of clauses

Randomized Algorithm

Simple Idea: Toss a coin, and independently set each variable to true with probability $1/2$

What is the expected number of clauses satisfied by a random assignment?

A random assignment to variables satisfies in expectation $7m/8$ clauses of a 3-CNF formula f with m clauses

Let Z_j be the random variable $Z_j = \begin{cases} 1 & \text{if clause } C_j \text{ is satisfied} \\ 0 & \text{otherwise} \end{cases}$

$E[Z_j] = Pr[C_j \text{ is satisfied}] = 1 - Pr[C_j \text{ is not satisfied}]$

C_j is not satisfied when all literals in C_j are set to FALSE (independently)

Thus, $Pr[C_j \text{ is not satisfied}] = (1/2)^3 = 1/8$ $\quad \triangleright \; E[Z_j] = 7/8$

Let Z be the number of clauses satisfied by the random assignment

$$E[Z] = \sum_{j=1}^{m} E[Z_j] = \sum_{j=1}^{m} \frac{7}{8} = \frac{7m}{8} \qquad \triangleright \text{ linearity of expectation}$$

For any instance of MAX-3-SAT with m clauses, there exists a truth assignment which satisfies at least $7m/8$ clauses

There is a non-zero probability that a random variable takes the value of its expectation

▷ Pigeon-hole principle of expectation

$$Pr\big[Z \geq E[Z]\big] > 0$$

Probabilistic Method:

Prove the existence of a non-obvious property by showing that a random construction produces it with positive probability

MAX-3-SAT Las Vegas ($7/8$)-Approximation

Is there a $7/8$ Las Vegas approximation algorithm for MAX-3-SAT?

- guaranteed to find an assignment satisfying at least $7m/8$ clauses
- expected runtime is polynomial

Standard trick: Repeatedly generate a random assignment A to variables until A satisfies at least $7m/8$ clauses

Suppose $Pr\left[A \text{ satisfies} \geq 7m/8 \text{ clauses}\right] \ \geq \ p$

Then, expected number of trials to find this assignment is $1/p$

$\triangleright$ Expectation of geometric random variable

If p is polynomial, then expected running time is polynomial

MAX-3-SAT Las Vegas (7/8)-Approximation

Probability p that a random assignment satisfies $\geq {}^{7m}/_{8}$ clauses is $\geq {}^{1}/_{8m}$

p_j : probability that the random assignment satisfies exactly j clauses

$\triangleright\ j = 1, 2, \cdots, m$

Lower bound on p using $E[Z] = {}^{7m}/_{8}$

$$E[Z] = \sum_{j=0}^{m} j\, p_j = \sum_{j < \frac{7m}{8}} j\, p_j + \sum_{j \geq \frac{7m}{8}} j\, p_j \leq \frac{7m-1}{8} \sum_{j < \frac{7m}{8}} p_j + m \sum_{j \geq \frac{7m}{8}} p_j$$

$$\implies E[Z] \leq \frac{7m-1}{8} \cdot 1 + m \cdot p \implies \frac{7m}{8} \leq \frac{7m-1}{8} + mp \implies p \geq \frac{1}{8m}$$

MAX-3-SAT cannot be approximated in polynomial time to within a ratio greater than 7/8, unless P=NP $\triangleright$ [Hástad 1997]

MAX-3-SAT: Derandomization

Random choices by an algorithm sometimes happen to be ‘good’

▷ i.e. the out the randomized algorithm is close to the optimal

Can these ‘good’ choices be made deterministically?

Derandomization: Transforming a randomized algorithm into a deterministic algorithm

Can the $7/8$-approx Las Vegas Algorithm for MAX-3-SAT be derandomized?

How do we know which set of choices for variable assignments is ‘good’? i.e. satisfies greater number of clauses

Idea: Consider the choice for each variable (True/False) one by one

Let Z be the number of clauses satisfied

Given assignments for the "first i" variables $x_1 = a_1 \cdots, x_i = a_i$, the expected value of Z with random assignment of the unassigned variables $x_{i+1}, \cdots, x_n$ can be computed in polynomial time

Given assignment to a variable, for each clause C_j if the corresponding literal evaluates to

- FALSE, then remove it from C_j
- TRUE, then ignore the clause as it is satisfied

Conditional expectation of Z is the unconditional expectation of Z in the reduced set of clauses plus the number of already satisfied clauses

This yields a polynomial time deterministic algorithm for MAX-3-SAT

Let Z be the number of clauses satisfied

1. Fix an order of variables $x_1, x_2, \cdots, x_n$
2. For $i = 1$ to n, If

$$E\left[Z | x_1 = a_1, \cdot\cdot, x_{i-1} = a_{i-1}, x_i = \text{TRUE}\right] > E\left[Z | x_1 = a_1, \cdot\cdot, x_{i-1} = a_{i-1}, x_i = \text{FALSE}\right]$$

   - then set x_i to TRUE
   - else set x_i to FALSE

- Since $E\left[Z | x_1 = a_1, \cdots, x_i = a_i\right] \geq E\left[Z\right]$ for $1 \leq i \leq n$
- And $E\left[Z\right] = {}^{7m}/_{8}$
- Thus, $E\left[Z | x_1 = a_1, \cdots, x_i = a_i\right] \geq {}^{7m}/_{8}$

Derandomized algorithm satisfies at least ${}^{7m}/_{8}$ clauses.