Randomized Computation

- Deterministic and (Las Vegas & Monte Carlo) Randomized Algorithms
- Probability Review
- Probabilistic Analysis of deterministic QUICK-SORT Algorithm
- RANDOMIZED-SELECT and RANDOMIZED-QUICK-SORT
- Max-Cut
- Min-Cut
- MAX-3-SAT and Derandomization
- Closest Pair
- Randomized Complexity Classes

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Probability Review

- Sample Space
- Event
- Random Variables
- Expectation, Linearity of Expectation, Indicator Random Variables
- Conditional Probability and Conditional Expectation
- Independence

Sample Space Ω

Sample Space Ω: Set of all possible outcome of a random experiments

- Toss a fair coin   $\Omega = \{H, T\}$
- Roll a fair die  $\Omega = \{1, 2, 3, 4, 5, 6\}$
- Roll two fair dice 

$$\Omega = \{(1,1),(1,2),(1,3)\cdots,(2,1),(2,2)\cdots,(6,6)\}$$

12	13	14	15	16
21	23	24	25	26
31	32	34	35	36
41	42	43	45	46
51	52	53	54	56
61	62	63	64	65

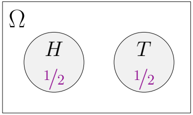
Mathematical Model of a random experiment

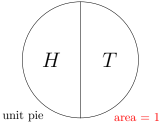
Sample Space Ω: Set of all possible outcome of a random experiments

Each outcome $i \in \Omega$ has probability $p(i)$ of occurring, such that

1. $Pr(i) \geq 0$
2. $\sum_{i \in \Omega} Pr(i) = 1$

Toss a fair coin 





What is the probability of a heads?

$Pr(\{H\}) = 1/2$

$Pr(\{H\}) + Pr(\{T\}) = 1/2 + 1/2 = 1$

Mathematical Model of a random experiment

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Each outcome $i \in \Omega$ has probability $p(i)$ of occurring, such that

1. $Pr(i) \ \geq \ 0$
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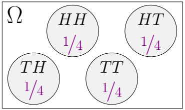
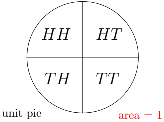
Toss two fair coins





What is the probability of a heads on both coins?

$Pr(\{HH\}) = 1/4$

$Pr(\{HH\}) + Pr(\{HT\}) + Pr(\{TH\}) + Pr(\{TT\}) \ = \ 1/4 + 1/4 + 1/4 + 1/4 = 1$

Mathematical Model of a random experiment

Sample Space Ω: Set of all possible outcome of a random experiments

Each outcome $i \in \Omega$ has probability $p(i)$ of occurring, such that

1. $Pr(i) \geq 0$
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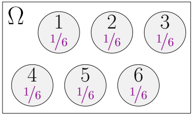
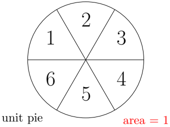
Roll a fair die 





What is the probability of rolling a 3?

$Pr(\{3\}) = 1/6$

$$\sum_{i=1}^{6} Pr(\{i\}) = \sum_{i=1}^{6} 1/6 = 1$$

Mathematical Model of a random experiment

Sample Space Ω: Set of all possible outcome of a random experiments

Each outcome $i \in \Omega$ has probability $p(i)$ of occurring, such that

$$Pr(i) \geq 0 \qquad \sum_{i \in \Omega} Pr(i) = 1$$

- Roll two fair dice 

11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
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61	62	63	64	65	66

$$\Omega = \{11, 12, 13 \cdots, 21, 22 \cdots, 66\}$$

What is the probability of rolling a 54?

$$Pr(\{54\}) = 1/36$$

$$\sum_{i=1}^{6} \sum_{j=1}^{6} Pr(\{ij\}) = \sum_{i=1}^{6} \sum_{j=1}^{6} 1/36 = 1$$

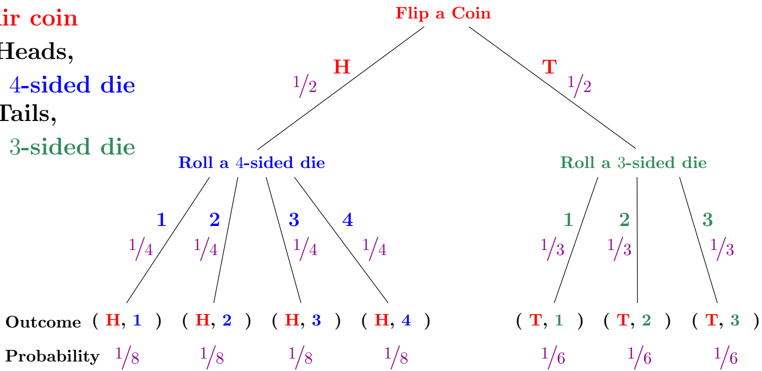
Mathematical Model of a random experiment

Sample Space Ω: Set of all possible outcome of a random experiments

Each outcome $i \in \Omega$ has probability $p(i)$ of occurring, such that

$$Pr(i) \geq 0 \qquad \sum_{i \in \Omega} Pr(i) = 1$$

Flip a fair coin
If it is Heads,
Roll a 4-sided die
If it is Tails,
Roll a 3-sided die



Events and their Probability

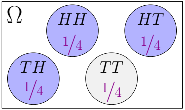
An event is S is a subset of the sample space, $S \subseteq \Omega$

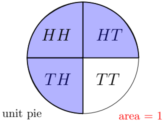
▷ Some group of outcomes we are interested in

Probability of the event S

$$Pr(S) = \sum_{i \in S} Pr(i)$$

Toss two fair coins 





- Event: At least one coin shows Heads

$$S \subseteq \Omega = \Big\{HH, HT, TH\Big\}$$

$$Pr(S) = Pr\left(\{HH, HT, TH\}\right) = 3 \times \frac{1}{4} = \frac{3}{4}$$

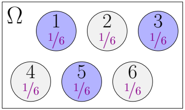
An event is S is a subset of the sample space, $S \subseteq \Omega$

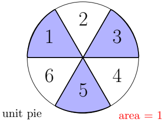
▷ Some group of outcomes we are interested in

Probability of the event S

$$Pr(S) = \sum_{i \in S} Pr(i)$$

Roll a fair die 





- Event: Value on the die is an odd number

$$O \subseteq \Omega = \{1, 3, 5\}$$

$Pr(O) = Pr(\{1, 3, 5\}) = 3 \times {}^1\!/_6 = {}^1\!/_2$

Events and their Probability

An event is S is a subset of the sample space, $S \subseteq \Omega$

▷ Some group of outcomes we are interested in

Probability of the event S

$$Pr(S) = \sum_{i \in S} Pr(i)$$

Roll two fair dice 

11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
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51	52	53	54	55	56
61	62	63	64	65	66

- Event: Sum of values on dice is 10 or more

$$S_1 \subseteq \Omega = \left\{46, 55, 56, 64, 65, 66\right\}$$

$$Pr(S_1) = Pr\left(\{46, 55, 56, 64, 65, 66\}\right) = 6 \times 1/36 = 1/6$$

Events and their Probability

An event is S is a subset of the sample space, $S \subseteq \Omega$

▷ Some group of outcomes we are interested in

Probability of the event S

$$Pr(S) = \sum_{i \in S} Pr(i)$$

Roll two fair dice 

11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

- Event: Both dice show an even number at least 4

$$S_2 \subseteq \Omega = \{44, 46, 64, 66\}$$

$$Pr(S_2) = Pr(\{44, 46, 64, 66\}) = 4 \times 1/36 = 1/9$$

Events and their Probability

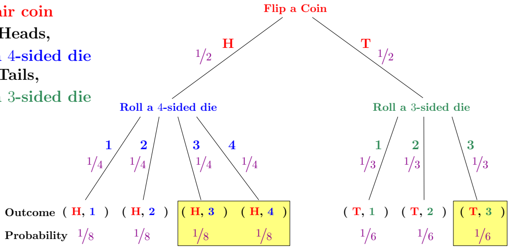
An event is S is a subset of the sample space, $S \subseteq \Omega$

▷ Some group of outcomes we are interested in

Probability of the event S

$$Pr(S) = \sum_{i \in S} Pr(i)$$

Flip a fair coin
If it is Heads,
Roll a 4-sided die
If it is Tails,
Roll a 3-sided die

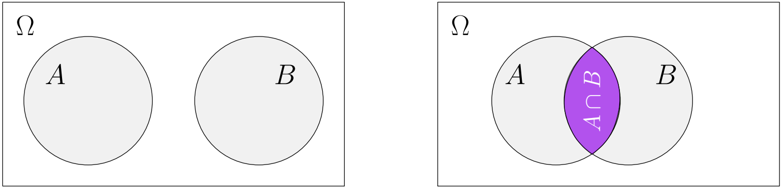


- Event: die roll is at least 3 $\qquad S \subseteq \Omega = \{H3, H4, T3\}$

$$Pr(S) = Pr(\{H3, H4, T3\}) = 1/8 + 1/8 + 1/6 = 5/12$$

Events and their Probability

For events A and B ,   find $Pr(A \cup B)$



Mutually exclusive events A and B Non-mutually exclusive events A and B

- If A and B are mutually exclusive $(A \cap B = \emptyset)$

$$Pr(A \cup B) = Pr(A) + Pr(B)$$

- If A and B are non-mutually exclusive $(A \cap B \neq \emptyset)$

$$Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$$

Conditional Probability

Conditional probability of an event X given that an event Y has occurred

Probability of X given Y, $Pr(X|Y)$

▷ Revised prob. of events conditioned on another event(partial info)

- Roll a fair die 
  - What is the probability that the outcome is 5?
  - Suppose the die rolls an even number, what is prob. that the outcome is 5?
  - Suppose the die rolls an odd number, what is prob. that the outcome is 5?

Conditional probability of an event X given that an event Y has occurred

Probability of X given Y, $\quad Pr(X|Y)$

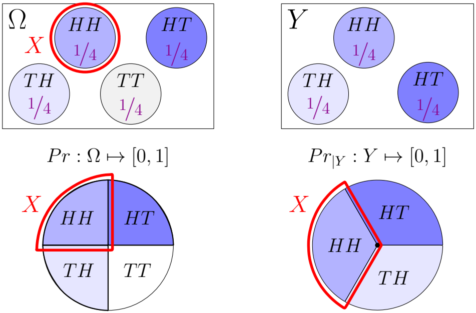
▷ Revised prob. of events conditioned on another event(partial info)

Toss two fair coins 

X: At least coin is Heads

Y: Both coins are Heads

- $Pr[X] = 1/4$
- $Pr[Y] = 3/4$
- $Pr[X|Y] = 1/3$



Conditional probability of an event X given that an event Y has occurred

Probability of X given Y, $\quad Pr(X|Y)$

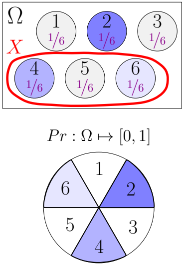
▷ Revised prob. of events conditioned on another event(partial info)

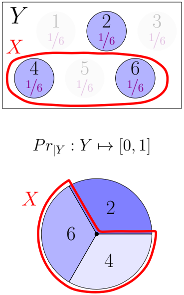
Roll a fair die 

X: Value on the die is ≥ 4

Y: Value on the die is even

- $Pr[X] = 1/2$
- $Pr[Y] = 1/2$
- $Pr[X|Y] = 2/3$





Conditional probability of an event X given that an event Y has occurred

Probability of X given Y, $\quad Pr(X|Y)$

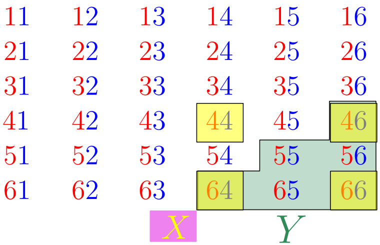
▷ Revised prob. of events conditioned on another event(partial info)

Roll two fair dice 

X: Sum of values on dice is ≥ 10

Y: Values on both dice are even ≥ 4

- $Pr[X] = 4/36$
- $Pr[Y] = 6/36$
- $Pr[X|Y] = 3/6$
- $Pr[Y|X] = 3/4$



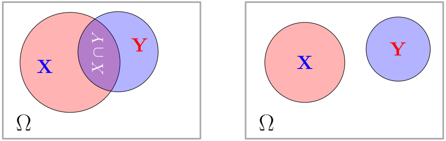
Conditional Probability

Conditional probability of an event X given that an event Y has occurred

Probability of X given Y, $Pr(X|Y)$

▷ Revised prob. of events conditioned on another event(partial info)

$$Pr(X|Y) = \frac{Pr(X \cap Y)}{Pr(Y)} \qquad Pr(Y|X) = \frac{Pr(X \cap Y)}{Pr(X)}$$



Independent Events

- Events A and B and independent if and only if

$$Pr(A \cap B) = Pr(A)Pr(B)$$

Equivalently

$$Pr(A|B) = Pr(A) \quad \text{and} \quad Pr(B|A) = Pr(B)$$

Random Variable

Random Variable: A function mapping outcomes to real numbers

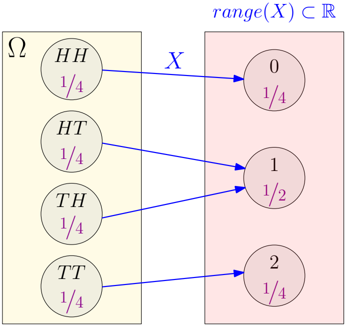
$$X : \Omega \mapsto \mathbb{R}$$

Numeric functions cannot be applied to events (being sets) but random variables can be manipulated

We can take mean, min, max, sum, product of random variables

Toss two fair coins 

$X : \Omega \mapsto \mathbb{R}$ = Number of Tails



Random Variable

Random Variable: A function mapping outcomes to real numbers

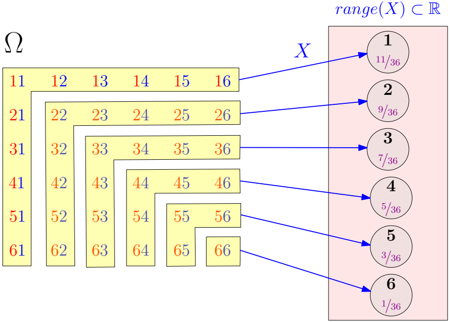
$$X : \Omega \mapsto \mathbb{R}$$

Numeric functions cannot be applied to events (being sets) but random variables can be manipulated

We can take mean, min, max, sum, product of random variables

Roll two fair dice 

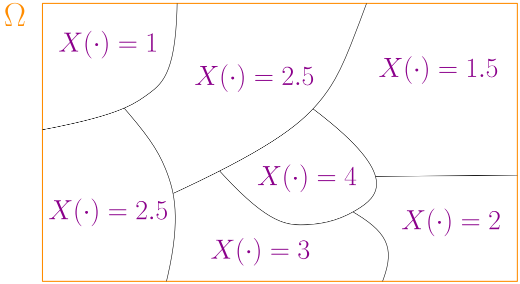
$X : \Omega \mapsto \mathbb{R}$ = Min of dice rolls



Probability of events from random variables

- $X : \Omega \mapsto \mathbb{R}$
- X partitions Ω into parts as pre-images of different values of X

$X : \Omega \mapsto \mathbb{R}$



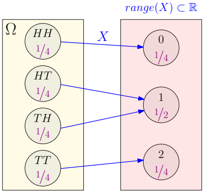
$$Pr(X = a) = Pr(\{w : X(w) = a\}) = \sum_{w : X(w) = a} Pr(w)$$

Expectation $E[X]$: The (weighted) average value of the random variable X

Simply average over every outcome in Ω weighted by its probability

$$E[X] \;=\; \sum_{i\in\Omega} X(i)\cdot Pr(i)$$

Toss two fair coins 



$X:\Omega \mapsto \mathbb{R}$ = Number of Tails

$$E(X) \;=\; \sum_{a=0}^{3} a \times Pr(X=a) \;=\; 0\times\frac{1}{4} + 1\times\frac{1}{2} + 2\times\frac{1}{4} \;=\; 1$$

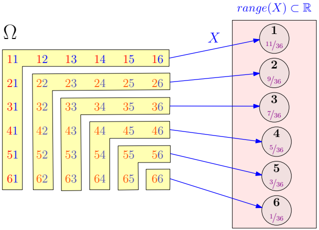
Expectation $E[X]$: The (weighted) average value of the random variable X

Simply average over every outcome in Ω weighted by its probability

$$E[X] = \sum_{i \in \Omega} X(i) \cdot Pr(i)$$

Roll two fair dice 

$X : \Omega \mapsto \mathbb{R}$ = Min of dice rolls



$$E(X) = \sum_{a=1}^{6} a \cdot Pr(X = a) = 1\frac{11}{36} + 2\frac{9}{36} + 3\frac{7}{36} + 4\frac{5}{36} + 5\frac{3}{36} + 6\frac{1}{36} = 2.527$$

Expectation $E[X]$: The (weighted) average value of the random variable X

Simply average over every outcome in Ω weighted by its probability

$$E[X] = \sum_{i \in \Omega} X(i) \cdot Pr(i)$$

Roll two fair dice  $\Omega = \{11, 12, 13 \cdots, 21, 22 \cdots, 66\}$

X = sum of the dice roll

$$E(X) = \sum_{a=1}^{6} \sum_{b=1}^{6} (a + b) Pr(X = a + b) = \sum_{a=1}^{6} \sum_{b=1}^{6} (a + b) \frac{1}{36} = 7$$

Linearity of Expectation

- $E(aX + b) \ = \ aE(X) + b$
- $E(aX + bY) \ = \ aE(X) + bE(Y)$
- Expectation of sum of random variables is sum of their expectations

$$E\left[\sum_{j=1}^{n} X_j\right] \ = \ \sum_{j=1}^{n} E\left[X_j\right]$$

Indicator Random Variables

Indicator Random Variable: A function that maps every outcome to 0 or 1

$X : \Omega \mapsto \{0, 1\}$ ▷ aka Bernoulli or characteristic random variable

- Identify all outcomes with and without a "characteristic"
- Partition the sample space into two parts

Roll two fair dice  $\Omega = \{11, 12, 13 \cdots, 21, 22 \cdots, 66\}$

$$Y = \begin{cases} 1 & \text{if sum of dice rolls is at least 10} \\ 0 & \text{else} \end{cases}$$

$$E(Y) = 1 \cdot p(Y = 1) + 0 \cdot p(Y = 0) = p(Y = 1)$$

Conditional Expectation

Expected value of a random variable X given that value of r.v Y is b

$$E[X|Y=b] \;=\; \sum_a a \cdot p(X=a|Y=b)$$

The average of values a's of X weighted by $p(X=a|Y=b)$

Roll two fair dice  $\Omega \;=\; \{11, 12, 13 \cdots, 21, 22 \cdots, 66\}$

- X = sum of the dice roll
- Y : indicator if the two dice show the same value

$$E[Y|X=10] = 1 \cdot p(Y=1|X=10) + 0 \cdot p(Y=0|X=10) = \frac{1}{3}$$

- $X = 10 \implies$  $\in \{46, 55, 64\}$
- $p(Y=1|X=10) = 1/3$ and $p(Y=0|X=10) = 2/3$

Conditional Expectation

Expected value of a random variable X given that value of r.v Y is b

$$E[X|Y=b] \;=\; \sum_{a} a \cdot p(X=a|Y=b)$$

The average of values a's of X weighted by $p(X=a|Y=b)$

Roll two fair dice  $\Omega \;=\; \{11, 12, 13 \cdots, 21, 22 \cdots, 66\}$

- $X =$ sum of the dice roll
- Y : indicator if the two dice show the same value

$$E[X|Y=1] = \sum_{a=2}^{12} a \cdot p(X=a|Y=1) = 7$$

- $Y=1 \implies$  $\in \{11, 22, 33, 44, 55, 66\}$
- $p(X=3|Y=1)=0$ and $p(X=4|Y=1)= 1/6$ $\ldots$

Conditional Expectation

$$E\left[E[Y|X]\right] = E[Y]$$

The expected value (over all possible values of X) of the expected value of Y conditioned on X is just the expected value of Y

$$E\left[E[X|Y]\right] = E[X]$$

Independent Random Variables

Random variables X and Y are independent if and only if

$$p(X = a \text{ AND } Y = b) = p(X = a)\, p(Y = b)$$

If X and Y are independent random variables, then

$$E[XY] = E[X]E[Y]$$