Finite Automata

- Deterministic Finite Automata
- Languages decided by a DFA – Regular Languages
- Closure Properties of regular languages
- Non-Deterministic Finite Automata, DFA= NFA
- Regular Expression: Computation as Description
- DFA=NFA=RegExp, Generalized NFA
- Non-Regular Languages, The Pumping Lemma
- Minimizing DFA

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Non-Regular Languages

The Problems Solved by DFAs

L is regular if there exists a DFA M, such that $L(M) = L$

Are all languages regular? No! Here are some non-regular languages

- BALANCED-STRINGS $= \{w | w$ has an equal number of 0's and 1's$\}$
- $0^n 1^n$
- $c^{251} a^n b^{2n}$
- $a^{2^n} \subseteq \{a\}^*$ (unary language)

How to tell if a language is not regular?

▷ Pumping Lemma and Myhill-Nerod theorem

How to tell if a language is regular?

▷ Give FA or Regular Expression, and Myhill-Nerod theorem

BALANCED-STRINGS is not a regular language

Formal proof later!

Need to prove that there is no DFA reognizing it

Intuitively,

- DFA must remember the frequencies of 0's and 1's seen so far
  - ▷ it can also remember the difference in the two frequencies
- The two frequencies (or their difference) is unbounded, there can't be enough states to keep track of
- Here the finite states/or constant memory is used critically

Pigeonhole Principle

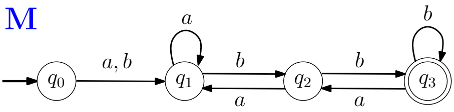
If there are more pigeons

than there are pigeon holes,

then some pigeon hole must have more than one pigeons

DFA walk on a string

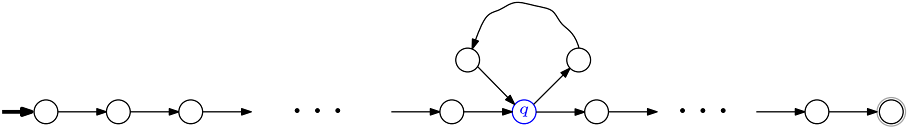
For a string $w \in \Sigma^*$, a w-**walk** or **walk on** w is the sequence of visited states when M is run on w



In walks on a, aa, ab, abb, bbb **no state is repeated**

In walks on aba, abba, aaba, abbb, bbbab **at least one state is repeated**

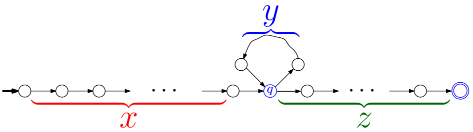
Generally, if string w has length $\geq$ number of states in M, then some state q must be repeated in the walk on w ▷ pigeon-hold principle

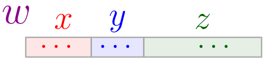


L : an infinite regular language, M : a DFA on m states recognizing L

Let w be a string such that $w \in L$ and $|w| \geq m$

Let q be the first state repeated in the walk on w





Let $w = xyz$ $|y| \geq 1$ $|xy| \leq m$ ▷ $\because$ q is the first state repeated

- $xz \in L$ ▷ accepting walk is present
- $xyz \in L$ ▷ $w \in L$
- $xyyz \in L$ ▷ by taking one more tour of the *'subwalk'* for y
- $xyyyz \in L$ ▷ by taking three tours of the *'subwalk'* for y
- $xy^iz \in L$, $i = 0, 1, 2 \ldots,$ ▷ by taking i tours of the *'subwalk'* for y

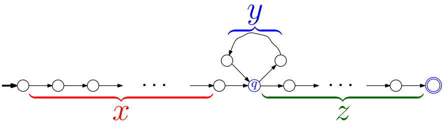
Pumping Lemma for Regular Languages

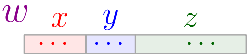
The Pumping Lemma

Let L be an infinite regular language.

There exists an integer m such that any string $w \in L$, with length $|w| \geq m$ can be written as $w = xyz$ with

- $|xy| \leq m$
- $|y| \geq 1$
- $xy^i z \in L$ for $i = 0, 1, 2, \ldots$





The y portion of the string w is pumped up

DFA Unsolvable Problems: $L = \{0^n 1^n : n \geq 0\}$

The language $L = \{0^n 1^n : n \geq 0\}$ is not regular

We use the pumping lemma to prove that L is not regular

Proof: Suppose L is a regular language. L is infinite

Let m be the integer guaranteed by the pumping lemma

Let $w = 0^m 1^m \in L$. By the pumping lemma w can be written as

$$w = 0^m 1^m = x\,y\,z = \overbrace{\underbrace{0\ldots00}_{x}\,\underbrace{0\ \ldots\ 0}_{y}\,0\ldots00}^{m}\overbrace{11\ldots\qquad\ldots11}^{m}$$

$$z = 0\ldots00\,11\ldots\qquad\ldots11$$

such that $|xy| \leq m$ and $|y| \geq 1 \implies y = 0^k\ ,\ 1 \leq k \leq m$

and $xy^2z = 0^{m+k}1^m \in L$ ▷ A contradiction

$$x\,y^2\,z = \overbrace{\underbrace{0\ldots00}_{x}\,\underbrace{0\ \ldots\ 0}_{y}\,\underbrace{0\ \ldots\ 0}_{y}\,0\ldots00}^{m+k}\overbrace{11\ldots\qquad\ldots11}^{m}$$

$$z = 0\ldots00\,11\ldots\qquad\ldots11$$

DFA Unsolvable Problems: $L = \{vv^R : v \in \{a,b\}^*\}$

The language $L = \{vv^R : v \in \{a,b\}^*\}$ is not regular

We use the pumping lemma to prove that L is not regular

Proof: Suppose L is a regular language. L is infinite

Let m be the integer guaranteed by the pumping lemma

Let $w = a^m b^m b^m a^m \in L$. By the pumping lemma w can be written as

$$w \;=\; x\,y\,z \;=\; \underbrace{a\ldots a}_{x}\underbrace{a\;\ldots\;a}_{y}\underbrace{a\ldots a b\ldots \ldots b\, b\ldots \ldots b a\ldots \ldots a}_{z}, \quad \overbrace{a\ldots a\,a\ldots a\,a\ldots a}^{m}\overbrace{b\ldots\ldots b}^{m}\overbrace{b\ldots\ldots b}^{m}\overbrace{a\ldots\ldots a}^{m}$$

such that $|xy| \le m$ and $|y| \ge 1 \implies y = a^k$, $1 \le k \le m$

and $\quad xy^2z \;=\; a^{m+k}b^m b^m a^m \;\in\; L$ $\quad\triangleright$ A contradiction

$$x\,y^2\,z \;=\; \underbrace{a\ldots a}_{x}\underbrace{a\;\ldots\;a}_{y}\underbrace{a\;\ldots\;a}_{y}\underbrace{a\ldots a b\ldots \ldots b\, b\ldots \ldots b a\ldots \ldots a}_{z}, \quad \overbrace{a\ldots a\,a\ldots a\,a\ldots a\,a\ldots a}^{m+k}\overbrace{b\ldots\ldots b}^{m}\overbrace{b\ldots\ldots b}^{m}\overbrace{a\ldots\ldots a}^{m}$$

DFA Unsolvable Problems: $L = \{a^p b^q c^{p+q} : p, q \geq 0\}$

The language $L = \{a^p b^q c^{p+q} : p, q \geq 0\}$ is not regular

We use the pumping lemma to prove that L is not regular

Proof: Suppose L is a regular language. L is infinite

Let m be the integer guaranteed by the pumping lemma

Let $w = a^m b^m c^{2m} \in L$. By the pumping lemma w can be written as

$$w = x\,y\,z = \overbrace{\underbrace{a \ldots a}_{x}\,\underbrace{a \ \ldots \ a}_{y}\,a \ldots a}^{m}\,\overbrace{b \ldots \qquad \ldots b}^{m}\,\overbrace{c \ldots \qquad \ldots cc \ldots \qquad \ldots c}^{2m}$$

where z spans from the $a \ldots a$ after y through the final c.

such that $|xy| \leq m$ and $|y| \geq 1 \implies y = a^k \ , \ 1 \leq k \leq m$

and $\quad xy^0z = xz = a^{m-k}b^m c^{2m} \in L$ $\qquad \triangleright$ A contradiction

$$x\,y^0\,z = \overbrace{\underbrace{a \ldots a}_{x}\,a \ldots a}^{m-k}\,\overbrace{b \ldots \qquad \ldots b}^{m}\,\overbrace{c \ldots \qquad \ldots cc \ldots \qquad \ldots c}^{2m}$$

where z spans from the $a \ldots a$ after x through the final c.

DFA Unsolvable Problems: $L = \{a^{n!} : n \geq 0\}$

The language $L = \{a^{n!} : n \geq 0\}$ is not regular

We use the pumping lemma to prove that L is not regular

Proof: Suppose L is a regular language. L is infinite

Let m be the integer guaranteed by the pumping lemma

Let $w = a^{m!} \in L$. By the pumping lemma w can be written as $w = xyz$ such that $|xy| \leq m$ and $|y| \geq 1 \implies y = a^k$, $1 \leq k \leq m$

and $xy^2z = a^{m+k+m!-m} = a^{m!+k} \in L$ $\triangleright$ A contradiction

Is $m! + k = p!$ for any integer k?

$m! < m! + k$ and $m! + k < (m+1)!$ for $k \leq m$