Computability Theory: Decidability and Recognizability

- Encoding Turing Machines and the Universal TM
- Computability
- Halt: Undecidable Problems using Diagnolization
- Accept: Undecidable Problems using Diagnolization
- Turing Reductions
- Mapping Reductions
- Undecidable and Unrecognizable Problems
- Rice Theorem

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Rice's Theorem

$L = \{\langle M, w\rangle :$ on input w, M tries to move head past the leftmost cell$\}$

L is undecidable

To prove L undecidable, we reduce A_{TM} (an undecidable problem) to it

$$A_{TM} \ \leq_m \ L$$

On input $\langle M, w\rangle$ to A_{TM}, design a TM N as follows:

- Initially, the tape of N contains a special symbol $\#$ followed by w
- N simulate M on w, if N's head hit $\#$ move it to right (M tries to move head past the left-most cell)
- If M accepts, N move its head past to left of $\#$

Define $f : A_{TM} \mapsto L$ as $f(\langle M, w\rangle) = \langle N, w\rangle$

$$\langle M, w\rangle \in A_{TM} \iff \langle N, w\rangle \in L$$

$L = \{\langle M, w\rangle :$ on input w M tries to moves head left at least once $\}$

L is decidable

To prove L decidable, we construct N to decide it

On input $\langle M, w\rangle$ to N, it works as follows:

- Run M on w for $|Q| + |w| + 1$ steps,
- If M ever moves it head left **Accept**

  else **Reject**

General Undecidability and Unrecognizability

Generally, we have seen analyzing programs (Turing machines) is hard

Rice's theorem makes the general statement of undecidability

Any non-trivial semantic property P of Turing machines is undecidable

Non-trivial: Not all TMs possess or lack the property

$\mathcal{M}_{P=Yes} = \{M : \text{ TM } M \text{ has the property } P\text{, i.e. } P(M) = Yes\}$
$\mathcal{M}_{P=No} = \{M : \text{ TM } M \text{ does not have the property } P\text{, i.e. } P(M) = No\}$

There are TMs with the property and there are TMs without the property

$$\mathcal{M}_{P=Yes} \neq \emptyset \qquad \mathcal{M}_{P=No} \neq \emptyset$$

Trivial properties examples: $L(M) \subset \Sigma^*, \quad |L(M)| \geq 0$

Semantic: The property relates to the behavior of TMs

For all TM M_1 and M_2, if $L(M_1) = L(M_2) \implies P(M_1) = P(M_2)$

Semantic and non-semantic properties

Semantic

- M accepts "010"
- M accepts $w \Leftrightarrow M$ accepts w^R
- $L(M) = \Sigma^*$
- $L(M) = \emptyset$
- $L(M)$ is regular
- $|L(M)| = 36$

Non-semantic

- M halts and reject "010"
- M moves its head left on input w
- M has 36 states
- M has ≥ 3 transitions from at least one state
- M reads ≤ 36 tapes cells
- M makes 36 transitions on "010"

Rice's Theorem

For a nontrivial semantic property $\mathbb{P}$, $L_{\mathbb{P}} = \{\langle M \rangle : \mathbb{P}(M) = \textbf{Yes}\}$ is undecidable

To prove $L_{\mathbb{P}}$ undecidable reduce A_{TM} (an undecidable problem) to it

Let N be a TM that accepts no string, i.e. $L(N) = \emptyset$

Since $\mathbb{P}$ is non-trivial, there exists a machine N', such that $\mathbb{P}(N') = \neg\mathbb{P}(N)$

Case 1: $\mathbb{P}(N) =$ **No** (and $\mathbb{P}(N') =$ **Yes**)

On input $\langle M, w \rangle$ to A_{TM}, define a TM M_w as follows

$M_w(x) :=$ If M accepts w and N' accepts x, then **Accept** else **Do not Accept**

M accepts $w \implies L(M_w) = L(N')$. $\mathbb{P}(N') = \textbf{Yes} = \mathbb{P}(M_w) \implies \langle M_w \rangle \in L_{\mathbb{P}}$

M does not accept $w \implies L(M_w) = \emptyset = L(N)$.
$\mathbb{P}(N) = \textbf{No} = \mathbb{P}(M_w) \implies \langle M_w \rangle \notin L_{\mathbb{P}}$

In either case we get answer to $\langle M, w \rangle \in A_{TM}$

Rice's theorem

Rice's Theorem

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To prove $L_{\mathbb{P}}$ undecidable reduce A_{TM} (an undecidable problem) to it

Let N be a TM that accepts no string, i.e. $L(N) = \emptyset$

Since $\mathbb{P}$ is non-trivial, there exists a machine N', such that $\mathbb{P}(N') = \neg\mathbb{P}(N)$

Case 2: $\neg\mathbb{P}(N) = \textbf{No}$ (and $\neg\mathbb{P}(N') = \textbf{Yes}$)

On input $\langle M, w \rangle$ to A_{TM}, define a TM M_w as follows

$M_w(x) :=$ If M accepts w and N' accepts x, then **Accept** else **Do not Accept**

M accepts $w \implies L(M_w) = L(N')$. $\neg\mathbb{P}(N') = \textbf{Yes} = \neg\mathbb{P}(M_w) \implies \langle M_w \rangle \in L_{\neg\mathbb{P}}$

M does not accept $w \implies L(M_w) = \emptyset = L(N)$.
$\neg\mathbb{P}(N) = \textbf{No} = \neg\mathbb{P}(M_w) \implies \langle M_w \rangle \notin L_{\neg\mathbb{P}}$

Note that this proves that $A_{TM} \leq_m L_{\neg\mathbb{P}} = \overline{L_{\mathbb{P}}}$

$\overline{L_{\mathbb{P}}}$ is undecidable $\iff$ $L_{\mathbb{P}}$ is undecidable