Computability Theory: Decidability and Recognizability

- Encoding Turing Machines and the Universal TM
- Computability
- Halt: Undecidable Problems using Diagnolization
- Accept: Undecidable Problems using Diagnolization
- Turing Reductions
- Mapping Reductions
- Undecidable and Unrecognizable Problems
- Rice Theorem

Imdad ullah Khan

More Computability Questions

More (meta) computational problems

$EQ_{DFA} = \{\langle D_1, D_2\rangle : D_1, D_2 \text{ are DFAs}, L(D_1) = L(D_2)\}$

$EQ_{REX} = \{\langle R_1, R_2\rangle : R_1, R_2 \text{ are Regexps}, L(R_1) = L(R_2)\}$

$\text{EMPTY}_{DFA} = \{\langle D\rangle : D \text{ is DFA}, L(D) = \emptyset\}$

$\text{EMPTY}_{NFA} = \{\langle N\rangle : N \text{ is NFA}, L(N) = \emptyset\}$

$\text{EMPTY}_{Rex} = \{\langle R\rangle : R \text{ is Regexp}, L(R) = \emptyset\}$

$\text{EMPTY}_{TM} = \{\langle M\rangle : M \text{ is TM}, L(M) = \emptyset\}$

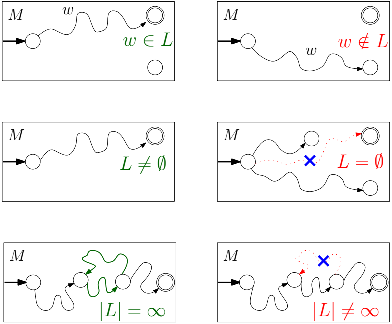
$EQ_{TM} = \{\langle D_1, M_2\rangle : M_1, M_2 \text{ are TMs}, L(M_1) = L(M_2)\}$

$\text{REGULAR}_{TM} = \{\langle M\rangle : M \text{ is TM}, L(M) \text{ IS REGULAR}\}$

Queries about Regular Languages

Given regular languages $L, L_1, L_2 \subset \Sigma^*$ and $L(M) = L$, $L(M_i) = L_i$

- Given $w \in \Sigma^*$, is $w \in L$?
  Run M on w and check if M accepts w
- Is $L = \emptyset$?
  Check if there is a path in M from start to final state
- Is L infinite?
  Check if there is (non-simple) walk in M from start to final state
- Given regular languages, Is $L_1 = L_2$?
  Check if $\left(L_1 \cap \overline{L_2}\right) \cup \left(\overline{L_1} \cap L_2\right) = \emptyset$



Try the questions $L = \Sigma^*$? $L_1 \cap L_2 = \emptyset$? $L_1 \subseteq L_2$? Is L finite?

Queries about Regular Languages

EMPTY_{DFA} is decidable

Since EMPTY_{DFA} is clearly decidable (just need to check reachability of final state(s))

Since for NFA and Regexp there is an equivalent DFA, we conclude that

EMPTY_{DFA} and EMPTY_{Rex} are also decidable

EQ_{DFA} is decidable

To check if $\langle D_1, D_2 \rangle \in EQ_{DFA} \iff L(D_1) = L(D_2)$
construct a DFA D such that $L(D) = \left(L_1 \cap \overline{L_2}\right) \cup \left(\overline{L_1} \cap L_2\right)$

Note that by closure properties of regular languages $L(D)$ is regular, hence DFA D exist

Next check if $\langle D \rangle \in \text{EMPTY}_{DFA}$

EQ_{Rex} is decidable

To prove that EQ_{Rex} is decidable reduce it to EQ_{DFA} (a decidable problem)

$EQ_{Rex} \leq_m EQ_{DFA}$

$f : EQ_{Rex} \mapsto EQ_{DFA}$ is a mapping reduction
Given $\langle R_1, R_2 \rangle$, construct DFAs D_1 and D_2 such that $L(D_1) = L(R_1)$ and $L(D_2) = L(R_2)$

$f(\langle R_1, R_2 \rangle) = \langle D_1, D_2 \rangle$

By equality of corresponding languages we get

$$\langle R_1, R_2 \rangle \in EQ_{Rex} \iff \langle D_1, D_2 \rangle \in EQ_{DFA}$$

Since EQ_{DFA} is decidable we get that EQ_{Rex} is also decidable

EMPTY_{TM} is unrecognizable

To prove EMPTY_{TM} unrecognizable reduce $\overline{A_{TM}}$ (:unrecognizable) to it

$\overline{A_{TM}} \leq_m \text{EMPTY}_{TM}$

Given $\langle M, w \rangle$, design a machine M' which outputs on w the same as M and rejects all other strings

i.e. On input x, M' works as, $M'(x) = \begin{cases} M(x) & \text{if } x = w \\ \textbf{reject} & \text{otherwise} \end{cases}$

Define $f : \overline{A_{TM}} \mapsto \text{EMPTY}_{TM}$ as $f(\langle M, w \rangle) = \langle M' \rangle$

Note $L(M') = \begin{cases} \{w\} & \text{if } M \text{ accepts } w \\ \emptyset & \text{else} \end{cases}$

$\langle M, w \rangle \notin A_{TM} \Leftrightarrow M$ does not accept $w \Leftrightarrow L(M') = \emptyset \Leftrightarrow \langle M' \rangle \in \text{EMPTY}_{TM}$

EQ_{TM} is unrecognizable

To prove EQ_{TM} unrecognizable reduce EMPTY_{TM} (:unrecognizable) to it

$\text{EMPTY}_{TM} \leq_m EQ_{TM}$

Let M' be a TM that rejects all strings ▷ e.g. having no accept state or no transition to it

$L(M') = \emptyset$

Define $f : \text{EMPTY}_{TM} \mapsto EQ_{TM}$ as $f(\langle M \rangle) = \langle M, M' \rangle$

$$\langle M \rangle \in \text{EMPTY}_{TM} \iff L(M) = L(M') \iff \langle M, M' \rangle \in EQ_{TM}$$

REGULAR_{TM} is unrecognizable

To prove REGULAR_{TM} unrecognizable reduce $\overline{A_{TM}}$ (:unrecognizable) to it

$\overline{A_{TM}} \leq_m \text{REGULAR}_{TM}$

Given $\langle M, w \rangle$, design a TM M' which outputs on x the same as M on w if $x = 0^n 1^n$ and rejects all other strings

i.e. On input x, M' works as, $\quad M'(x) = \begin{cases} M(w) & \text{if } x = 0^n1^n \\ \textbf{reject} & \text{otherwise} \end{cases}$

Define $f : \overline{A_{TM}} \mapsto \text{REGULAR}_{TM}$ as $f(\langle M, w \rangle) = \langle M' \rangle$

- $\langle M, w \rangle \in A_{TM} \implies f(\langle M, w \rangle) = M'$ such that M' accepts 0^n1^n
- $\langle M, w \rangle \notin A_{TM} \implies f(\langle M, w \rangle) = M'$ such that M' accepts nothing

In the prior case, $L(M')$ is regular and in the latter case it is not

$$\langle M, w \rangle \in \overline{A_{TM}} \iff M' \in \text{REGULAR}_{TM}$$