Computability Theory: Decidability and Recognizability

- Encoding Turing Machines and the Universal TM
- Computability
- Undecidable Problems using Diagnolization
- The Halting and Accept Problems
- Turing Reductions
- Mapping Reductions
- Undecidable and Unrecognizable Problems
- Rice Theorem

IMDAD ULLAH KHAN

The Halting Problem

A concrete Undecidable Problem

We showed existence of unrecognizable and undecidable languages.

May be only weired langauges are undecidable!

We show undecidable and unrecognizable languages that are very interesting (natural, useful)

$\text{HALT}_{TM} = \{\langle M, w\rangle : M \text{ a TM over } \Sigma,\ M \text{ halts on } w \in \Sigma^*\}$

How to check if a program halts

Algorithm Clearly halts

return true

Algorithm Halts if n is even and $n \geq 0$

while $n \neq 0$ **do**
  $n \leftarrow n - 2$

Algorithm Never halts

$n \leftarrow 1$
while $n \neq 0$ **do**
  $n \leftarrow n + 1$

The Collatz Conjecture (1937)

Algorithm Collatz Program(integer n)

while $n \neq 1$ **do**
  if n is even **then**
    $n \leftarrow n/2$
  else
    $n \leftarrow 3n + 1$

On input $n = 3$, we get $3, 10, 5, 15, 8, 4, 2, 1$

On input $n = 4$, we get $4, 2, 1$

On input $n = 5$, we get $5, 16, 8, 4, 2, 1$

On input $n = 6$, we get $6, 3, 10, 5, 16, 8, 4, 2, 1$

On input $n = 7$, we get $7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 15, 8, 4, 2, 1$

On input $n = 8$, we get $8, 4, 2, 1$

On input $n = 9$, we get $9, 28, 14, 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 15, 8, 4, 2, 1$

27, 82, 41, 124, 62, 31, 94, 47, 142, 71, 214, 107, 322, 161, 484, 242, 121, 364, 182, 91, 274, 137, 412, 206, 103, 310, 155, 466, 233, 700, 350, 175, 526, 263, 790, 395, 1186, 593, 1780, 890, 445, 1336, 668, 334, 167, 502, 251, 754, 377, 1132, 566, 283, 850, 425, 1276, 638, 319, 958, 479, 1438, 719, 2158, 1079, 3238, 1619, 4858, 2429, 7288, 3644, 1822, 911, 2734, 1367, 4102, 2051, 6154, 3077, 9232, 4616, 2308, 1154, 577, 1732, 866, 433, 1300, 650, 325, 976, 488, 244, 122, 61, 184, 92, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1

The Collatz Conjecture (1937)

Algorithm Collatz Program(integer n)

while $n \neq 1$ **do**
  if n is even **then**
    $n \leftarrow n/2$
  else
    $n \leftarrow 3n + 1$

Collatz Conjecture (1937)

For every integer n this program eventually reaches 1 (thus halts)

aka wondrous numbers, $3n + 1$ conjecture, Syracuse problem, Ulam conjecture

> **For about a month everyone at Yale worked on it, with no result. A similar phenomenon happened when I mentioned it at the University of Chicago. A joke was made that this problem was part of a conspiracy to slow down mathematical research in the U.S.**
>
> **Shizuo Kakutani, 1960**

> **Mathematics is not yet ripe enough for such questions.**
>
> **Paul Erdös, 1983**

$\text{NegFLT}(n) \in \text{HALT} \iff$ Fermat's last theorem is false

Fermat's Last Theorem (1637)

For $n \geq 3$, $a^n + b^n = c^n$ has no solution where a, b, c are positive integers

Algorithm NegFLT(integer n)

```
flag ← true
a ← 1
while flag = true do
    for b = 1 → a do
        for c = 2 → a + b do
            if a^n + b^n = c^n then
                flag ← false
    a ← a + 1
```

This algorithm (TM) halts if and only if Fermat's Last Theorem is false

An algorithm for $\text{HALT}(\langle M, w \rangle)$ would yield a proof or disproof for FLT

Ok! we know FLT is true, how about some other

$\textsc{NegGoldbach}() \in \textsc{halt} \iff$ Goldbach conjecture is false

Goldbach Conjecture (1742)

Every even integer $n > 2$ is the sum of two primes.

Algorithm $\textsc{NegGoldbach}($ even integer $n)$

```
flag ← true
n ← 2
while flag = true do
    flag ← false
    n ← n + 2
    for p = 2 → n do
        if IsPrime(p) AND IsPrime(n − p) then
            flag ← true
            break
```

This algorithm (TM) halts if and only if Goldbach Conjecture is false

An algorithm for $\textsc{halt}(\langle M, w\rangle)$ would resolve the Goldbach conjecture

The Halting Problem is undecidable

$\text{HALT} = \{\langle M, w\rangle : \ M \text{ halts on input } w\}$ is undecidable

Proof: Suppose HALT is decidable and M_H is the decider TM

$$M_H(\langle M, w\rangle) = \begin{cases} \textbf{accept} & \text{if } M \text{ halts on } w \\ \textbf{reject} & \text{if } M \text{ does not halt on } w \end{cases}$$

Define a TM D that takes input encoding of a TM as follows:

$D(\langle M\rangle)$:

Run M_H on $\langle M, M\rangle$ and

$$D(\langle M\rangle) = \begin{cases} \textbf{loop} & \text{if } M_H \text{ accepts } \langle M, M\rangle \\ \textbf{halt} & \text{if } M_H \text{ rejects } \langle M, M\rangle \end{cases}$$

$D(\langle M\rangle)$ **loops if** M **halts on** $\langle M\rangle$

$D(\langle M\rangle)$ **halts if** M **loops on** $\langle M\rangle$

What if we run D on $\langle D\rangle$?

$$D(\langle D\rangle) = \begin{cases} \textbf{loops} & \text{if } D \text{ halts on} \langle D\rangle \\ \textbf{halts} & \text{if } D \text{ loops on} \langle D\rangle \end{cases}$$

The Halting Problem is undecidable

HALT $= \{\langle M, w\rangle :\ M$ halts on input $w\}$ is undecidable

The set of all TM's is countable, so we can make the table

Partial Output Table of $M_H(\langle M, w\rangle)$

	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$	$\langle M_5\rangle$	$\cdots$	$\langle D\rangle$	$\cdots$
M_1	**halts**	**halts**	**loops**	**loops**	**loops**		**halts**	
M_2								
M_3								
M_4								
M_5								
$\vdots$								
D								
$\vdots$								

The Halting Problem is undecidable

$\text{HALT} = \{\langle M, w\rangle : M \text{ halts on input } w\}$ is undecidable

The set of all TM's is countable, so we can make the table

Partial Output Table of $M_H(\langle M, w\rangle)$

	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$	$\langle M_5\rangle$	$\cdots$	$\langle D\rangle$	$\cdots$
M_1	halts	halts	loops	loops	loops		halts	
M_2	halts	loops	loops	halts	loops		halts	
M_3	loops	halts	halts	loops	halts		loops	
M_4	halts	loops	halts	halts	loops		loops	
M_5	loops	halts	loops	halts	loops		halts	
$\vdots$								
D								
$\vdots$								

The Halting Problem is undecidable

$\text{HALT} = \{\langle M, w\rangle : M$ halts on input $w\}$ is undecidable

The set of all TM's is countable, so we can make the table

Partial Output Table of $M_H(\langle M, w\rangle)$

	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$	$\langle M_5\rangle$	$\cdots$	$\langle D\rangle$	$\cdots$
M_1	halts	halts	loops	loops	loops		halts	
M_2	halts	loops	loops	halts	loops		halts	
M_3	loops	halts	halts	loops	halts		loops	
M_4	halts	loops	halts	halts	loops		loops	
M_5	loops	halts	loops	halts	loops		halts	
$\vdots$								
D	loops							
$\vdots$								

The Halting Problem is undecidable

$\text{HALT} = \{\langle M, w\rangle : M \text{ halts on input } w\}$ is undecidable

The set of all TM's is countable, so we can make the table

Partial Output Table of $M_H(\langle M, w\rangle)$

	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$	$\langle M_5\rangle$	$\cdots$	$\langle D\rangle$	$\cdots$
M_1	halts	halts	loops	loops	loops		halts	
M_2	halts	loops	loops	halts	loops		halts	
M_3	loops	halts	halts	loops	halts		loops	
M_4	halts	loops	halts	halts	loops		loops	
M_5	loops	halts	loops	halts	loops		halts	
$\vdots$								
D	loops	halts						
$\vdots$								

The Halting Problem is undecidable

$\text{HALT} = \{\langle M, w\rangle : M \text{ halts on input } w\}$ is undecidable

The set of all TM's is countable, so we can make the table

Partial Output Table of $M_H(\langle M, w\rangle)$

	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$	$\langle M_5\rangle$	$\cdots$	$\langle D\rangle$	$\cdots$
M_1	halts	halts	loops	loops	loops		halts	
M_2	halts	loops	loops	halts	loops		halts	
M_3	loops	halts	halts	loops	halts		loops	
M_4	halts	loops	halts	halts	loops		loops	
M_5	loops	halts	loops	halts	loops		halts	
$\vdots$						$\ddots$		
D	loops	halts	loops	loops	halts			
$\vdots$								

The Halting Problem is undecidable

HALT $= \{\langle M, w\rangle : M$ halts on input $w\}$ is undecidable

The set of all TM's is countable, so we can make the table

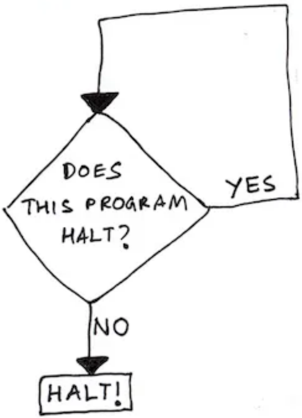
Can D be in the list? What will be its diagonal entry?

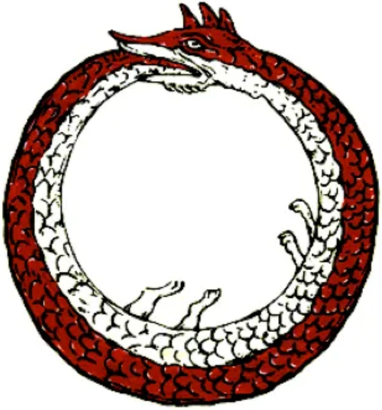
Partial Output Table of $M_H(\langle M, w\rangle)$

	$\langle M_1\rangle$	$\langle M_2\rangle$	$\langle M_3\rangle$	$\langle M_4\rangle$	$\langle M_5\rangle$	$\cdots$	$\langle D\rangle$	$\cdots$
M_1	halts	halts	loops	loops	loops		halts	
M_2	halts	loops	loops	halts	loops		halts	
M_3	loops	halts	halts	loops	halts		loops	
M_4	halts	loops	halts	halts	loops		loops	
M_5	loops	halts	loops	halts	loops		halts	
$\vdots$						$\ddots$		
D	loops	halts	loops	loops	halts		?	
$\vdots$								

D halts on $\langle D\rangle$ iff D loops on $\langle D\rangle$

The Halting Problem is undecidable





source: wired.com