Computability Theory: Decidability and Recognizability

- Encoding Turing Machines and the Universal TM
- Computability
- Undecidable Problems using Diagnolization
- The Halting and Accept Problems
- Turing Reductions
- Mapping Reductions
- Undecidable and Unrecognizable Problems
- Rice Theorem

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Computability Questions

Languages about computation

Some meta-computational problems (problems about computations)

- $A_{DFA} = \{\langle D, w\rangle : D \text{ a DFA over } \Sigma,\ D \text{ accepts } w \in \Sigma^* \leftrightarrow w \in L(D)\}$
- $A_{NFA} = \{\langle N, w\rangle : N \text{ a NFA over } \Sigma,\ N \text{ accepts } w \in \Sigma^* \leftrightarrow w \in L(N)\}$
- $A_{TM} = \{\langle M, w\rangle : M \text{ a TM over } \Sigma,\ M \text{ accepts } w \in \Sigma^*\}$
- $\text{HALT}_{TM} = \{\langle M, w\rangle : M \text{ a TM over } \Sigma,\ M \text{ halts on } w \in \Sigma^*\}$

Universal Turing Machine

Theorem

There is a Turing machine U that takes as input an encoding of an arbitrary Turing machine M over Σ and a string $w \in \Sigma^*$ such that U accepts $\langle M, w \rangle$ if and only if M accepts w

In other words, the language A_{TM} is Turing-recognizable.

There is no Universal DFA/NFA

In other words, A_{DFA} and A_{NFA} are not regular

A_{DFA} is decidable

$A_{DFA} = \{\langle D, w\rangle : D \text{ a DFA over } \Sigma,\ D \text{ accepts } w \in \Sigma^* \leftrightarrow w \in L(D)\}$

DFA is a special case of a Turing Machine.

Run the universal TM U on $\langle D, w\rangle$ and output the answer of D on w

A_{NFA} is decidable

$A_{NFA} = \{\langle N, w\rangle : N \text{ an NFA over } \Sigma,\ N \text{ accepts } w \in \Sigma^* \leftrightarrow w \in L(N)\}$

$NFA = DFA$ is a special case of a Turing Machine.

Run the universal TM U on $\langle N, w\rangle$ and output the answer of N on w

Is A_{TM} decidable?

$A_{TM} = \{\langle M, w \rangle : M \text{ a TM over } \Sigma, \; M \text{ accepts } w \in \Sigma^*\}$

Run the universal TM U on $\langle M, w \rangle$ and output the answer of M on w

What if M loops forever?

Is A_{TM} recognizable?

$A_{TM} = \{\langle M, w\rangle : M \text{ a TM over } \Sigma,\ M \text{ accepts } w \in \Sigma^*\}$

Run the universal TM U on $\langle M, w\rangle$ and accept if M accepts w

What if M loops forever?

Is HALT_{TM} decidable?

$\text{HALT}_{TM} = \{\langle M, w\rangle : M \text{ a TM over } \Sigma,\ M \text{ halts on } w \in \Sigma^*\}$

Run the universal TM U on $\langle M, w\rangle$ and accept if M does halt and reject otherwise

What if M loops forever?

Is HALT_{TM} recognizable?

$\text{HALT}_{TM} = \{\langle M, w\rangle : M \text{ a TM over } \Sigma,\ M \text{ halts on } w \in \Sigma^*\}$

Run the universal TM U on $\langle M, w\rangle$ and accept if M does halt

What if M loops forever?

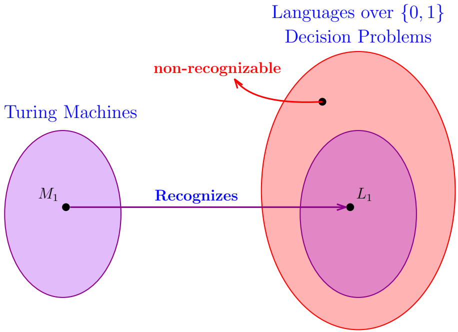
Existence of non-recognizable languages

There are non-recognizable languages

Assuming Church-Turing thesis, this means there are problems that no computing device can ever solve ▷ Non-Computable Problems

We prove this first result of computability theory by a counting argument

We show that there are more problems than there are Turing machines



Existence of non-recognizable languages

There are more problems than there are Turing machines

For any set A there is no onto function from A to $\mathcal{P}(A)$

Proof: Suppose $f : A \mapsto \mathcal{P}(A)$ is an onto function.

Note that for $x \in A$, $f(x) \in \mathcal{P}(A)$, i.e. $f(x) \subseteq A$

Define $S \subset A := \{x \in A : x \notin f(x)\}$

Since f is onto, the set S has a pre-image i.e. for some $x \in A$, $f(x) = S$

- If $x \in S$, then $x \notin f(x) = S$
- If $x \notin S$, then $x \in f(x) = S$

$\therefore \quad \forall x \in A, \ f(x) \neq S$, meaning f is not onto

No matter what the set A is, $\mathcal{P}(A)$ has strictly larger cardinality than A

Existence of non-recognizable languages

There are more problems than there are Turing machines

For any set A there is no onto function from A to $\mathcal{P}(A)$

Let $\mathcal{M}$ be the set of all Turing machines

$\mathcal{M} \subset \{0,1\}^* = B$

Let $\mathcal{L}$ be the set of all languages over $\{0,1\}$

since a language is a subset of B, we get $\mathcal{L} = \mathcal{P}(B)$

Suppose every language is recognizable, i.e.

$$\forall L \in \mathcal{L}, \quad \exists M \in \mathcal{M} \quad \text{such that} \quad L(M) = L$$

The mapping $R : \mathcal{M} \mapsto \mathcal{L}$, such that $R(M) = L(M)$ is an onto function $\triangleright$ a contradiction