Asymptotic Analysis

- Runtime Analysis
- Big Oh - $O(\cdot)$
- Complexity Classes and Curse of Exponential Time
- $\Omega(\cdot)$, $\Theta(\cdot)$, $o(\cdot)$, $\omega(\cdot)$ - Relational properties

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Asymptotic Lower Bound

Definition (Ω (Big Omega))

A function $g(n) \in \Omega(f(n))$ if there exists constant $c > 0$ and $n_0 \geq 0$ such that

$$g(n) \geq c(f(n)) \qquad \forall n \geq n_0$$

- Written as: $g(n) = \Omega(f(n))$
- $f(n)$ is an asymptotic lower bounded for $g(n)$
- $g(n) \in O(f(n)) \Leftrightarrow f(n) \in \Omega(g(n))$
- The definition of Ω works just like $O(\cdot)$, except that the function $g(n)$ is bounded from below, rather than from above
- A notion of $a \geq b$ for functions as for real numbers

Big Omega: Example

1. $3n^2 + 4n + 5 \in \Omega(n^2)$
2. $3n^2 + 4n + 5 \in \Omega(n)$
3. $3n^2 + 4n + 5 \neq \Omega(n^3)$

Asymptotic Tight Bounds

Definition (Θ (Big Theta))

A function $g(n)$ is $\Theta(n)$ iff there exists two positive real constants c_1 and c_2 and a positive integer n_0 such that $c_1 f(n) \leq g(n) \leq c_2 f(n)$ $\forall n > n_0$.
$n_0 = max[n_1, n_2]$

- $g(n) \in O(f(n))$ and $g(n) \in \Omega(f(n)) \Leftrightarrow g(n) \in \Theta(f(n))$
- Asymptotically tight bounds on worst-case running times characterize the performance of an algorithm precisely up to constant factors

Asymptotic Tight Bounds - Big Θ Notation

- $f(n) = pn^2 + qn + r$

$$f(n) \in \Omega(n^2), \quad \text{and} \quad f(n) \in O(n^2) \implies f(n) \in \Theta(n^2)$$

▷ p, q, r are positive constants

- $3n^2 + 4n + 5 \in \Theta(n^2)$
- $3n^2 + 4n + 5 \notin \Theta(n^3)$
- $3n^2 + 4n + 5 \notin \Theta(n)$

Little Oh - o Notation

Definition

A function $g(n) \in o(f(n))$ if for every constant $c > 0$, there exists a constant $n_0 \geq 0$ such that

$$g(n) \leq cf(n) \qquad \forall\ n \geq n_0$$

- Written as: $g(n) \in o(f(n))$
- This is used to show that g grows much much slower than f
- $f(n) \in o(g(n)) \Leftrightarrow (f(n) \in O(g(n)) \wedge f(n) \notin \Theta(g(n)))$
- An equivalent formulation (when $f(n)$ is non-zero) is given as $\lim_{n\to\infty} \dfrac{g(n)}{f(n)} = 0$

Little Oh - Examples

1. $3n^2 + 4n + 5 \notin o(n^2)$
2. $3n^2 + 4n + 5 \in o(n^3)$
3. $3n^2 + 4n + 5 \notin o(n)$

Little omega - ω Notation

Definition

A function $g(n) \in \omega((f(n))$ if for every constant $c > 0$, there exists constant $n_0 \geq 0$ such that

$$g(n) \geq cf(n) \qquad \forall\ n \geq n_0$$

- Written as: $g(n) \in \omega(f(n))$
- In this case f grows much faster than g.
- $f(n) \in \omega(g(n)) \Leftrightarrow (f(n) \in \Omega(g(n)) \wedge f(n) \notin \Theta(g(n)))$

Little omega: Examples

1. $3n^2 + 4n + 5 \notin \omega(n^2)$
2. $3n^2 + 4n + 5 \in \omega(n)$
3. $3n^2 + 4n + 5 \notin \omega(n^3)$

Properties of Asymptotic Growth Rates

Many relational properties of real numbers apply to asymptotic comparisons.

For the following, assume that f and g are asymptotically positive.

Transitivity

1. If $f \in O(g)$ and $g \in O(h)$, then $f \in O(h)$
2. If $f \in \Omega(g)$ and $g \in \Omega(h)$, then $f \in \Omega(h)$
3. if $f \in \Theta(g)$ and $g \in \Theta(h)$, then $f \in \Theta(h)$

Properties of Asymptotic Growth Rates

Reflexivity

1. $f \in O(f)$
2. $f \in \Omega(f)$
3. $f \in \Theta(f)$

Additivity

- If $f \in O(h)$ and $g \in O(h)$, then $f + g \in O(h)$
- In general, for constant k, if $f_1, f_2, \ldots, f_k$ and h are functions such that $f_i \in O(h)$ for all i. Then $f_1 + f_2 + \ldots + f_k \in O(h)$

Properties of Asymptotic Growth Rates

Symmetry

1. $f \in \Theta(g)$ if and only if $g \in \Theta(h)$

Transport Symmetry

1. $f \in O(g)$ if and only if $g \in \Omega(f)$
2. $f \in o(g)$ if and only if $g \in \omega(f)$