Network Flow

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Max Flow : Problem Formulation

Input: A flow network $G = (V, E, c)$, $c : E \to \mathbb{R}^+$

$s \in V$ a source and $t \in V$ a sink

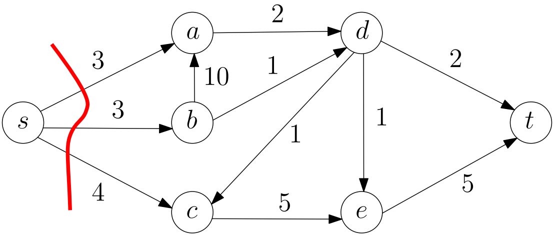
$f : E \to \mathbb{R}^+$ ($f_e = f(e)$) is a flow if it satisfies

1. capacity constraints $\forall\, e \in E \;:\; 0 \leq f_e \leq c_e$
2. flow conservation constraints $\forall\, v \in V\,, v \neq s, t \quad f^{out}(v) = f^{in}(v)$

$$size(f) \;=\; f^{out}(s) \;=\; f^{in}(t)$$

Output: A flow f of maximum size

Max Flow : Upper Bound

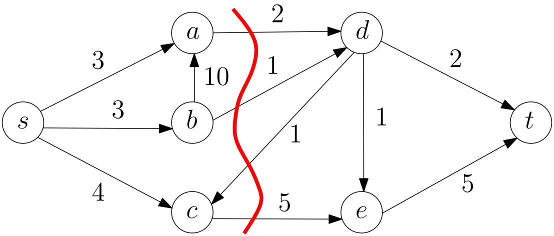


Consider the cut $\left[\{s\}, \overline{\{s\}}\right]$

Any flow generated from s has to go through one of the cut edges

Hence no flow can be of size bigger than $3 + 3 + 4 = 10$

Max Flow : Upper Bound



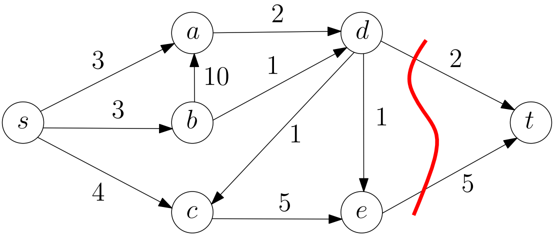
The same is true for nay cut, Consider the cut $\left[\{s, a, b, c\}, \{d, e, t\}\right]$

Any flow generated from s has to go through one of the cut edges

Hence no flow can be of size bigger than $2 + 1 + 5 = 8$

This is a tighter bound than the one we got from $\left[\{s\}, \overline{\{s\}}\right]$

Max Flow : Upper Bound



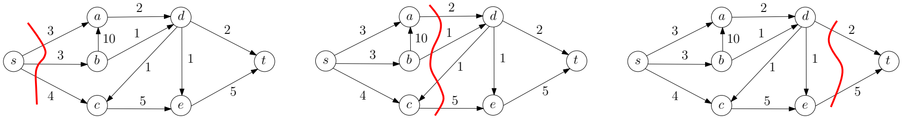
The same is true for nay cut, Consider the cut $\left[\overline{\{t\}}, \{t\}\right]$

Any flow generated from s has to go through one of the cut edges

Hence no flow can be of size bigger than $2 + 5 = 7$

This is a tighter bound than the one we got from $\left[\{s, a, b, c\}, \{d, e, t\}\right]$

Max Flow : Upper Bound



All cuts have s on one side and t on the other side

$s-t$ cut

$A \subset V$, an $s-t$ cut, $\left[A, \overline{A}\right]$, is a cut in G with $s \in A$ and $t \in \overline{A}$ content...

Capacity of an $s-t$ cut: sum of capacities of edges going from A to $\overline{A}$

$$c\left(\left[A, \overline{A}\right]\right) \;=\; \sum_{e \text{ outgoing from } A} c_e$$

Max Flow : Upper Bound

Let f be a flow in G and let $[A, \overline{A}]$ be any $s-t$ cut in G, then

$$size(f) \leq c([A, \overline{A}])$$

Proof: Let $[A, \overline{A}]$ be any cut. By definition we know that

$$size(f) = f^{out}(s) = f^{out}(s) + \sum_{s \neq v \in A} \left(f^{out}(v) - f^{in}(v)\right) \quad \text{just adding 0's}$$

$$= f^{out}(s) + \sum_{s \neq v \in A} f^{out}(v) - \sum_{s \neq v \in A} f^{in}(v)$$

$$= f^{out}(A) - f^{in}(A)$$

$$= \sum_{e \text{ outgoing from } A} f_e - \sum_{e \text{ incoming to } A} f_e \quad \text{flows on other edges cancel}$$

$$\leq \sum_{e \text{ outgoing from } A} c_e - \sum_{e \text{ incoming to } A} f_e \leq \sum_{e \text{ outgoing from } A} c_e = c([A, \overline{A}])$$

Max Flow : Upper Bound

Let f be a flow in G and let $\left[A, \overline{A}\right]$ be any $s-t$ cut in G, then

$$size(f) \leq c\left(\left[A, \overline{A}\right]\right)$$

Tightest upper bound will come from a $s-t$ cut of minimum capacity

$\left[A^*, \overline{A^*}\right]$ be an $s-t$ cut with minimum capacity ▷ min-$s-t$-cut

We get the corollary

$$size(f) \leq c\left(\left[A^*, \overline{A^*}\right]\right)$$