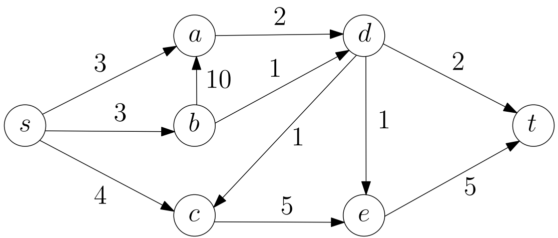
Network Flow

- Maximum Flow: Problem Formulation
- Maximum Flow: Upper Bound
- Maximum Flow: Adding flow along paths
- Residual Network and Augmenting Path
- Ford-Fulkerson Algorithm – Max-Flow-Min-Cut Theorem
- Edmond-Karp Algorithm
- Maximum Flow: Variants and Applications

Imdad ullah Khan

Network Flows : Motivation

A network of pipelines along which oil can be sent



We want to ship as much oil from s to t as possible

There are two restrictions

- A pipeline cannot carry more oil than weight of the corresponding edge
- No node can store any oil

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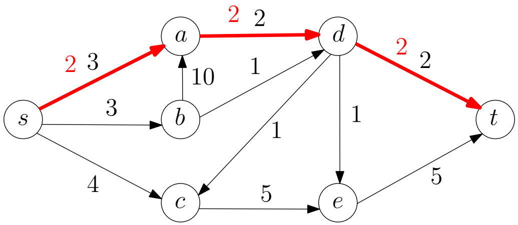

Send 2 units of flows along the path s, a, d, t

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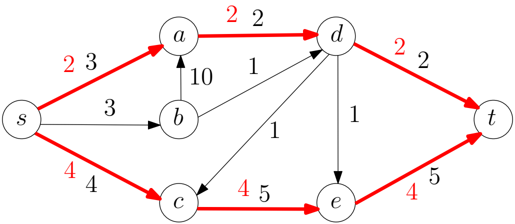

Send 2 units of flows along the path s, a, d, t

Another 4 units of flow can go through the path s, c, e, t

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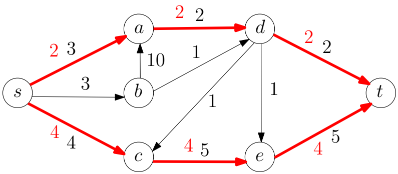


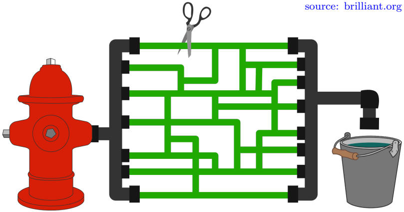
- Is this the best flow (the largest amount of oil that can be shipped)?
- How do we measure the size of flow?
- How do we determine that a given flow is the maximum possible?

Max Flow : Problem Formulation

A flow network: A directed graph with weights on edges

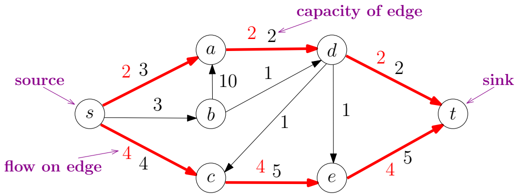
- Models a transportation network
- Edges can carry traffic and nodes switch traffic between edges
  - highway network vertices:intersections/interchanges, edges: roads
  - communication network vertices: switches/routers, edges: links
  - fluid networks vertices: junctures where pipes are plugged, edges: pipelines
- Edge weights are capacities of links (max traffic they can carry)

source: brilliant.org



Max Flow : Problem Formulation

A flow network: A directed graph with weights on edges



- capacity of the edge uv: $e = uv$ is associated with $c_e = c_{uv} \in \mathbb{R}^+$
- source s: $deg^-(s) = 0$ is the traffic generator
- sink t: $deg^+(t) = 0$ is the traffic consumer
- A $s - t$ flow: assignment to each edge e a flow $f_e \in \mathbb{R}^+$ with $f_e \leq c_e$
- flow $f : E \to \mathbb{R}^+$ satisfying the capacity and storage constraints

Max Flow : Problem Formulation

Given a flow network $G = (V, E, c)$, $c : E \to \mathbb{R}^+$

$s \in V$ a source and $t \in V$ a sink

$f : E \to \mathbb{R}^+$ $\quad (f_e = f(e))$ $\quad$ is a flow if it satisfies

1. (capacity constraints):

$$\forall\, e \in E \;:\quad 0 \;\leq\; f_e \;\leq\; c_e$$

2. (flow conservation constraints):

$$\forall\, v \in V\,, v \neq s, t \;:\quad \underbrace{\sum_{e \text{ into } v} f_e}_{\substack{\text{total flow} \\ \text{incoming to } v}} \;=\; \underbrace{\sum_{e \text{ out of } v} f_e}_{\substack{\text{total flow} \\ \text{outgoing from } v}}$$

Max Flow : Problem Formulation

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$s \in V$ a source and $t \in V$ a sink

$f : E \to \mathbb{R}^+$ $\quad (f_e = f(e))$ $\quad$ is a flow

$$\text{For } v \in V, \quad f^{out}(v) := \sum_{\substack{e \text{ outgoing} \\ \text{from } v}} f_e \quad \text{and} \quad f^{in}(v) := \sum_{\substack{e \text{ incoming} \\ \text{to } v}} f_e$$

$$\text{For } X \subset V, \quad f^{out}(X) := \sum_{\substack{e \text{ outgoing} \\ \text{from } X}} f_e \quad \text{and} \quad f^{in}(X) := \sum_{\substack{e \text{ incoming} \\ \text{to } X}} f_e$$

Max Flow : Problem Formulation

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$s \in V$ a source and $t \in V$ a sink

$f : E \to \mathbb{R}^+$ $\quad (f_e = f(e))$ $\quad$ is a flow if it satisfies

1. capacity constraints $\quad \forall\, e \in E \;:\; 0 \leq f_e \leq c_e$
2. flow conservation constraints $\quad \forall\, v \in V\,, v \neq s, t \quad f^{out}(v) = f^{in}(v)$

The size of flow f is total flow sent out from s

$$size(f) = \sum_{\substack{e \text{ outgoing} \\ \text{from } s}} f_e = f^{out}(s)$$

$$size(f) = f^{out}(s) = f^{in}(t)$$

Max Flow : Problem Formulation

Input: A flow network $G = (V, E, c)$, $c : E \to \mathbb{R}^+$

$s \in V$ a source and $t \in V$ a sink

$f : E \to \mathbb{R}^+$ ($f_e = f(e)$) is a flow if it satisfies

1. capacity constraints $\quad \forall\, e \in E \;:\; 0 \leq f_e \leq c_e$
2. flow conservation constraints $\quad \forall\, v \in V\,, v \neq s, t \quad f^{out}(v) = f^{in}(v)$

$$size(f) \;=\; f^{out}(s) \;=\; f^{in}(t)$$

Output: A flow f of maximum size