

## Minimum Spanning Tree

- Minimum Spanning Tree
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  - Basic Implementation
  - Vertex-Centric Implementation
  - Heap Based Implementation

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# Prim's Algorithm

**Input:** A weighted graph  $G = (V, E, w)$ ,  $w : E \rightarrow \mathbb{R}$

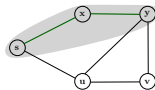
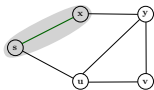
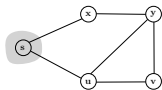
**Output:** A spanning tree of  $G$  with minimum total weight

## The Prim's algorithm

- Maintains a set  $R \subset V$  and a tree  $T$  spanning vertices in  $R$
- Initially  $R = \{s\}$ , an arbitrary vertex and  $T = \emptyset$
- Grow  $R$  by adding one vertex  $v$  in every iteration
- Grow  $T$  by adding an edge connecting  $v$  to some vertex in current
- $V(T) = R$  (vertices spanned by  $T$ )
- Select a minimum crossing edge from  $R$  to  $\bar{R}$

$$\arg \min_{e=(u,v), u \in R, v \notin R} w(e)$$

- Add  $v$  to  $R$  and  $e$  to  $T$



# Prim's Algorithm

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**Algorithm** Prim's Algorithm for MST in  $G = (V, E, w)$

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$R \leftarrow \{s\}$

▷  $s \in V$  an arbitrary vertex

$T \leftarrow \emptyset$

▷ Begin with an empty tree

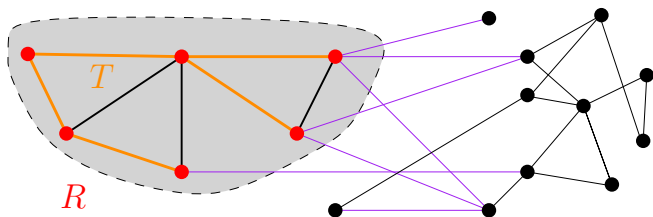
**while**  $R \neq V$  **do**

    Get  $e = (u, v)$ ,  $u \in R, v \notin R$  with minimum  $w(uv)$

$T \leftarrow T \cup \{e\}$

$R \leftarrow R \cup \{v\}$

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# Cuts in Graphs

A cut in  $G$  is a subset  $S \subset V$

- Denoted as  $[S, \bar{S}]$ ,  $S = \emptyset$  and  $S = V$  are trivial cuts
- An edge  $(u, v)$  is **crossing the cut**  $[S, \bar{S}]$ , if  $u \in S$  and  $v \in \bar{S}$
- **Empty Cut Lemma:**
  - A graph  $G$  is disconnected iff it has a cut with no crossing edge
- **Double Crossing Lemma**
  - If a cycle crosses a cut, then it has to cross at least twice
- **Lonely Crossing Lemma**
  - If  $e$  is the only edge crossing a cut  $[S, \bar{S}]$ , then it is not in any cycle
- **The Blue Rule**
  - If an edge  $e \in E$  is the lightest edge crossing some cut  $[S, \bar{S}]$ , then  $e$  belongs to the MST of  $G$

## Prim's Algorithm: Correctness and Optimality

**Algorithm** Prim's Algorithm for MST in  $G = (V, E, w)$

$R \leftarrow \{s\}$

▷  $s \in V$  an arbitrary vertex

$T \leftarrow \emptyset$

▷ Begin with an empty tree

**while**  $R \neq V$  **do**

    Get  $e = (u, v)$ ,  $u \in R, v \notin R$  with minimum  $w(uv)$

$T \leftarrow T \cup \{e\}$

$R \leftarrow R \cup \{v\}$

**Correctness:**  $T$  is a spanning tree of  $G$

- $T$  is a subgraph of  $G$
- $T$  is connected
- $T$  has no cycle
- $T$  is spanning

**Optimality:**  $T$  is the minimum spanning tree of  $G$

# Prim's Algorithm: Correctness and Optimality

**Correctness:**  $T$  is a spanning tree of  $G$

After every iteration  $i$ ,  $T$  is a spanning tree of  $G|_R$

**Proof:** by induction on  $|R|$  (iteration  $i$ )

- $|R| = |\{s\}| = 1$ , and  $T = \emptyset$  is a spanning tree of  $\{s\}$
- If  $T$  spans  $R$ ,  $|R| = i$ , then in the next iteration, we add a vertex to  $R$  with an edge connecting it to some vertex in  $R$ , hence  $T$  is a spanning tree of  $G|_R$
- $T$  has at most  $n - 1$  edges (max number of iterations)
- $T$  has at least  $n - 1$  edges
- If we can't add an edge from  $R \neq V$ , then  $[R, \bar{R}]$  is an empty cut
- Every time we add one edge from  $[R, \bar{R}]$  so  $T$  has no cycle

## Prim's Algorithm: Correctness and Optimality

**Optimality:**  $T$  is the minimum spanning tree of  $G$

Proof follows from the cut property

If an edge  $e \in E$  is the lightest edge crossing some cut  $[S, \bar{S}]$ , then  $e$  belongs to the MST of  $G$

- In every iteration we added the lightest edge crossing the cut  $[R, \bar{R}]$
- The blue rule guarantees this to be part of the MST
- Hence, by the blue rule, the output  $T$  is an optimal spanning tree