Minimum Spanning Tree

- Minimum Spanning Tree
- Prim's Algorithm for MST
- Cuts in Graphs
- Correctness and Optimality of Prim's Algorithm
- Runtime
  - Basic Implementation
  - Vertex-Centric Implementation
  - Heap Based Implementation

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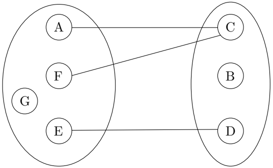
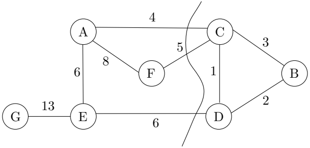
Cuts in Graphs

- Cuts in graphs are useful structures, helps analyzing MST algorithm
- We will discuss it in network flows, complexity, randomized algorithms

A cut in G is a subset $S \subset V$

- Denoted as $[S, \overline{S}]$
  - ▷ $S = \emptyset$ and $S = V$ are trivial cuts, we assume that $\emptyset \neq S \neq V$
- A graph on n vertices has 2^n cuts
- An edge (u, v) is crossing the cut $[S, \overline{S}]$, if $u \in S$ and $v \in \overline{S}$



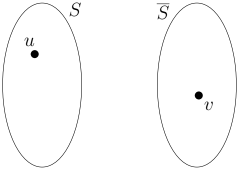


Empty Cut Lemma

A graph G is disconnected iff it has a cut with no crossing edge (empty cut)

Proof: if part

- Let $[S, \overline{S}]$ be an empty cut
- Let $u \in S$ and $v \in \overline{S}$
- No crossing edge implies no path between u and v



Empty Cut Lemma

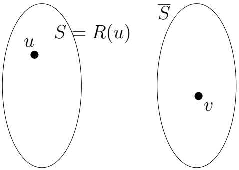
A graph G is disconnected iff it has a cut with no crossing edge (empty cut)

Proof: only if part

- Let u and v be disconnected
- Let $S = R(u)$ (vertices reachable from u)

$\triangleright$ S is the connected component containing u

- No edge crosses the cut $[S, \overline{S}]$
- Otherwise the endpoint of crossing edge must be in S

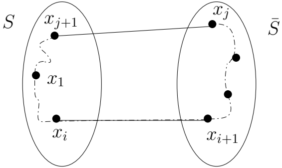


Double and Lonely Crossing Lemma

If a cycle crosses a cut, then it has to cross at least twice

A edge-subset (subgraph) crossing a cut means there is an edge crossing the cut

- A cycle starting in S once reaches $\overline{S}$ must have another edge to come back to S
- Actually any cycle must cross a cut an even number times



If e is the only edge crossing a cut $[S, \overline{S}]$, then it is not in any cycle

The cut property (Blue Rule)

If an edge $e \in E$ is the lightest edge crossing some cut $[S, \overline{S}]$, then e belongs to the MST of G

This statement assume edge weights are unique. More generally,

If an edge $e \in E$ is a lightest edge crossing some cut $[S, \overline{S}]$, then e belongs to some MST of G

Proof of the cut property (blue rule)

If an edge $e \in E$ is the lightest edge crossing some cut $[S, \overline{S}]$, then e belongs to the MST of G

Proof by contradiction:

- Let T^* be the "MST" of G
- Let e be lightest edge across a cut $[S, \overline{S}]$
- Suppose $e \notin T^*$
- There must some edge $f \in T^*$ across $[S, \overline{S}]$
  - $\because$ otherwise T^* is not connected, hence not a tree
- Exchange e with $f \in T^*$ to get T'
- $w(T') \leq w(T^*)$ as $w(e) < w(f)$
- Is T' a spanning tree?

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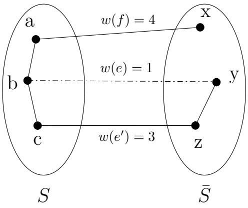
- Is T' a spanning tree?



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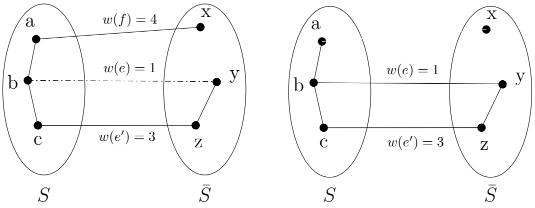
- Exchange e with $f \in T^*$ to get T', $w(T') \leq w(T^*)$ as $w(e) < w(f)$
- Replacing an arbitrary heavier crossing edge by e does not work



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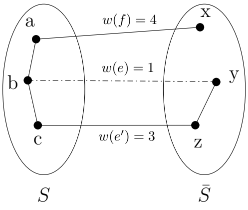
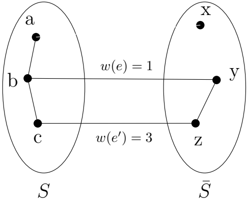
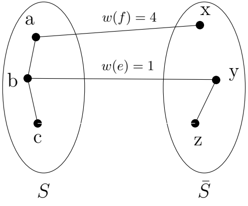
- Suppose $e \notin T^* \implies$ there must some edge $f \in T^*$ across $[S, \overline{S}]$
- Exchange e with $f \in T^*$ to get T', $w(T') \leq w(T^*)$ as $w(e) < w(f)$
- Replacing an arbitrary heavier crossing edge by e does not work
- Which edge should e replace?







Proof of the cut property (blue rule)

If an edge $e \in E$ is the lightest edge crossing some cut $[S, \overline{S}]$, then e belongs to the MST of G

Proof by contradiction:

- Let e be lightest edge across a cut $[S, \overline{S}]$
- Suppose $e \notin T^*$, the MST of G
- Add e to T^*
- It must create a cycle ▷ (a tree is maximally acyclic)
- The cycle must cross the cut at least twice ▷ (double crossing lemma)
- Let e' be another crossing edge on that cycle
- $T' = T^* \setminus \{e'\} \cup \{e\}$ and $w(T') < w(T^*)$