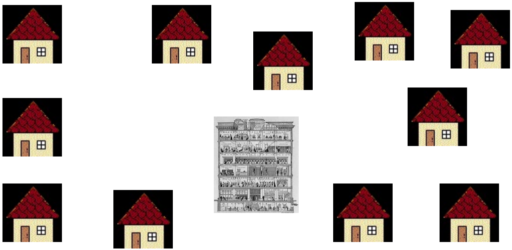
Minimum Spanning Tree

- Minimum Spanning Tree
- Prim's Algorithm for MST
- Cuts in Graphs
- Correctness and Optimality of Prim's Algorithm
- Runtime
  - Basic Implementation
  - Vertex-Centric Implementation
  - Heap Based Implementation

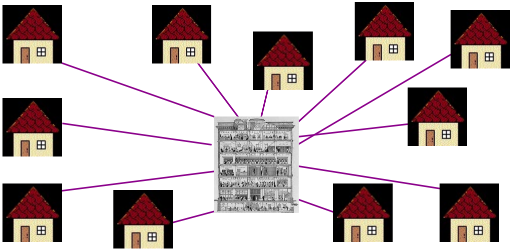
Imdad ullah Khan

Optimal connection between points



- Layout a telephone network so as every user is connected to all others
- Goal is to use as little wire as possible
- The network should not have any cycle (should be a tree)

Optimal connection between points



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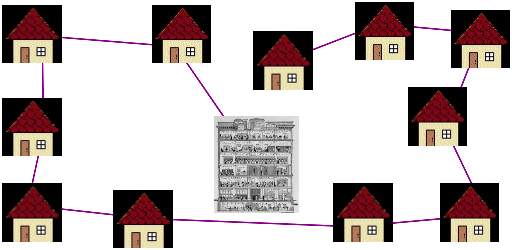
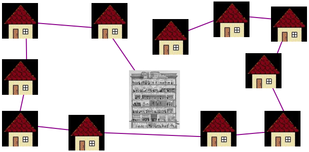


- Layout a telephone network so as every user is connected to all others
- Goal is to use as little wire as possible
- The network should not have any cycle (should be a tree)
- Naive approach (star network) may use a lot of wires
- Many possible solutions

Minimum Spanning Tree



- Layout a telephone network so as every user is connected to all others
- Goal is to use as little wire as possible

- In this problem all pairwise connections are possible
- The underlying graph is a complete graph
- Weight of edges are lengths of physical paths between nodes
- There could be restrictions on possible edges (not complete graphs)
- Weight of edges could be arbitrary

Minimum Spanning Tree Problem

Input: A weighted graph $G = (V, E, w)$, $w : E \to \mathbb{R}$

Output: A spanning tree of G with minimum total weight

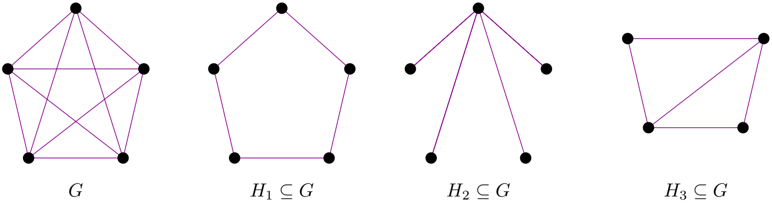
Weight of a tree T is sum of weights of its edges $w(T) = \sum_{e \in T} w(e)$

Subgraph

$H = (V', E')$ is a subgraph of $G = (V, E)$ if

- $V' \subseteq V$
- $E' \subseteq E$
- $E' \subseteq \binom{V'}{2}$

Denoted as $H \subseteq G$



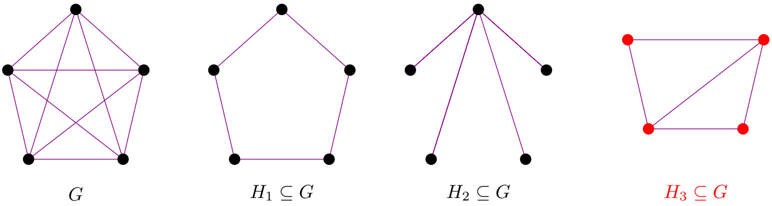
H_1, H_2, and H_3 are subgraphs of G

Spanning Subgraph

$H = (V', E')$ is a spanning subgraph of $G = (V, E)$ if

- $V' = V$
- $E' \subseteq E$

Denoted as $H \subseteq G$

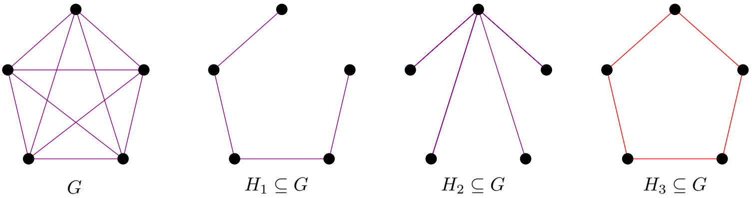


H_1 and H_2 are spanning subgraphs of G, while H_3 is not

Spanning Tree of Graphs

$H = (V', E')$ is a spanning tree of $G = (V, E)$ if

- H is a spanning subgraph of G
- H is a tree



H_1 and H_2 are spanning trees of G, while H_3 is not

Basic Facts about Tree

A connected graph on n vertices has at least $n - 1$ edges

A graph on n vertices and $\geq n$ edges has a cycle

A tree on n vertices has $n - 1$ edges

In a tree every pair of vertices has a unique path between them

A tree is a minimally connected graph

▷ Removing any edge from a tree disconnects it

A tree is a maximally acyclic graph

▷ Adding any edge to a tree creates a cycle

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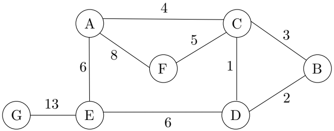
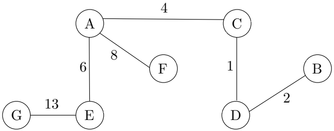
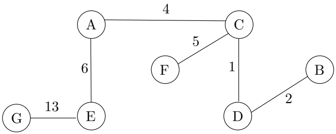


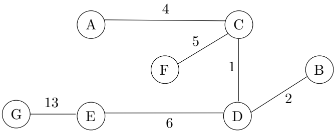
A weighted graph G



A spanning tree of G with weight 34



An MST of G with weight 31



An MST of G with weight 31

MST does not have to be unique