Basic Graph Algorithms

- Exploring Graphs
- Depth First Search
- DFS Forest - Start and Finish Time
- DAG, Topological Sorting
- Strongly Connected Components
- Breadth First Search
- Bipartite Graphs

IMDAD ULLAH KHAN

Depth First Search: Forest

Algorithm DFS(G)

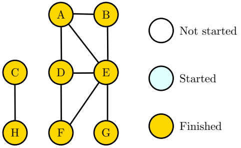
$visited \leftarrow$ ZEROS(n)
for all $v \in V$ **do**
  if $visited[v] = 0$ **then**
    DFS-EXPLORE(v)

Algorithm DFS-EXPLORE(s)

function DFS-EXPLORE(s)
  $visited[s] \leftarrow 1$
  for $u \in N(s)$ **do**
    if $visited[u] = 0$ **then**
      DFS-EXPLORE(u)

When call to DFS-EXPLORE(s) is executed, all vertices in $R(s)$ are visited

- When DFS-EXPLORE(u) is finished one 'DFS *tree*' is formed containing all vertices reachable from u
- The next DFS-EXPLORE(v) called from outer loop forms a new tree



DFS: Review

Algorithm DFS(G)

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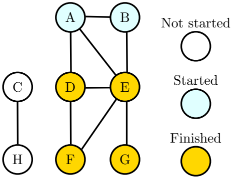
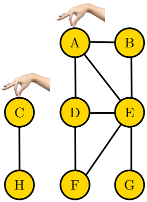
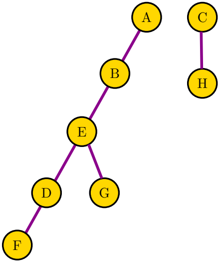
- DFS explores the entire graph
- Explores one neighbor first (in depth), before going to next neighbor
- Works both for undirected and directed graphs
- Runtime of $O(n + m)$ means DFS doesn't add any cost (asymptotically) to any graph algorithm, we typically do it as a pre-processing step
- Answers many questions
  - is graph connected, how many components in the graph, find $R(s)$, $\cdots$
- A fundamental algorithm has many applications we will discuss some
- For applications we need some extra book-keeping with DFS

DFS Forest



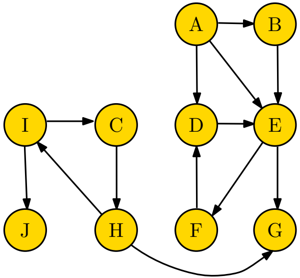


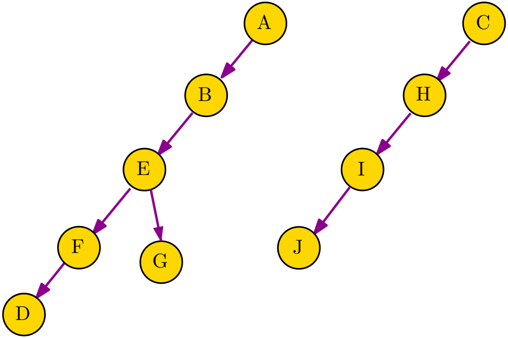


- Record predecessor relationships (save call hierarchy)
- Implicitly build a forest
- Predecessors subgraph (edges used for calling) forms DFS forest
- We first go as deep as we can
- For undirected graphs, each DFS tree is a connected component

DFS Forest: Directed Graphs

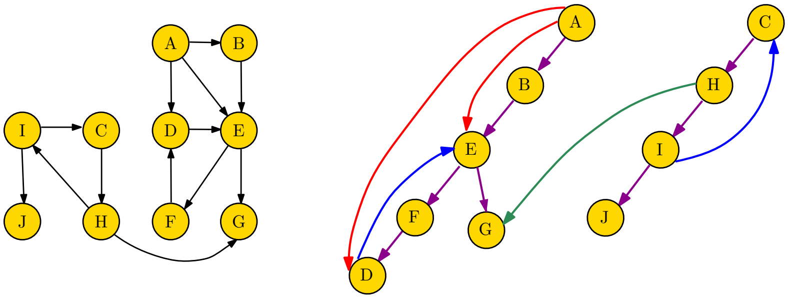
DFS Forest of a digraph





DFS Forest: Types of Edges

Overlay all edges of a digraph G onto its DFS forest



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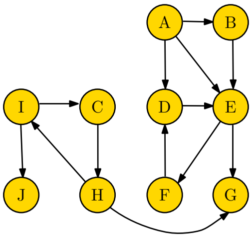

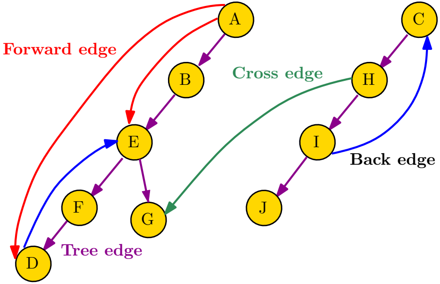

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- Back edge - Edge from a node to a non-parent ancestor
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DFS Forest: Start and Finish Exploration

- Extra book keeping: timestamps for each vertex
- **start time**: $s[v]$ and **end time**: $f[v]$

Algorithm DFS(G)

```
visited ← ZEROS(n)
time ← 1
for all v ∈ V do
    if visited[v] = 0 then
        DFS-EXPLORE(v)
```

Algorithm DFS-EXPLORE

```
function DFS-EXPLORE(v)
    visited[v] ← 1
    s[v] ← time
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    for u ∈ N(v) do
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DFS Forest: Start and Finish Time

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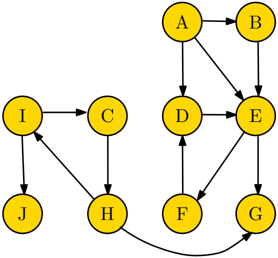

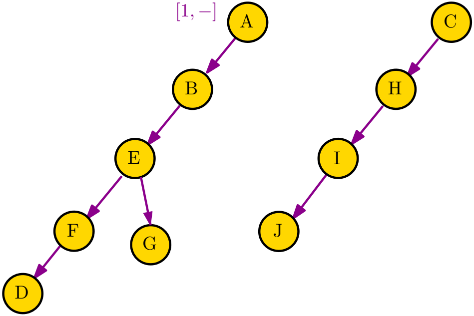

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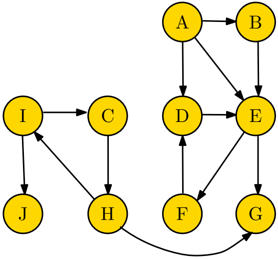

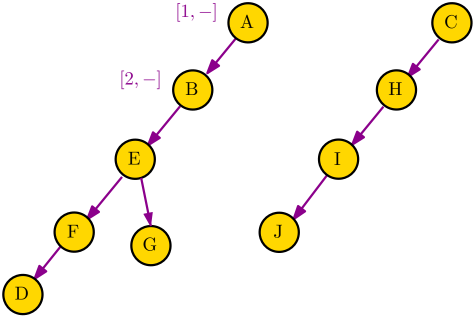

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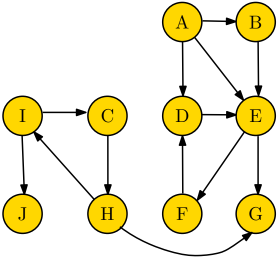

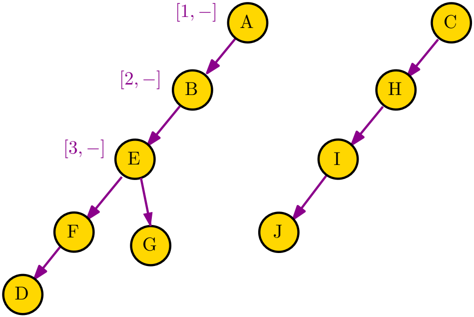

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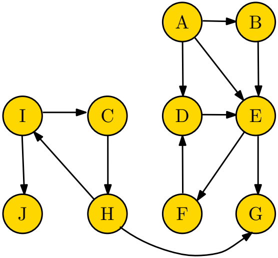

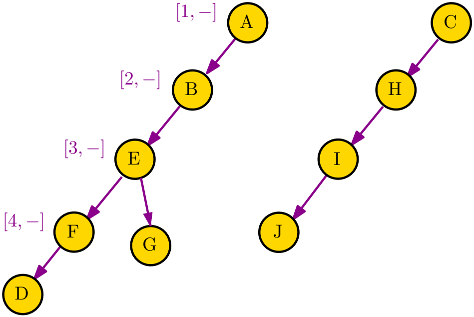

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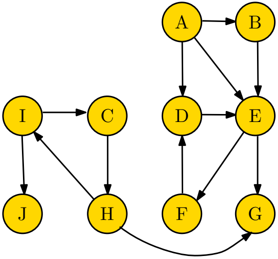

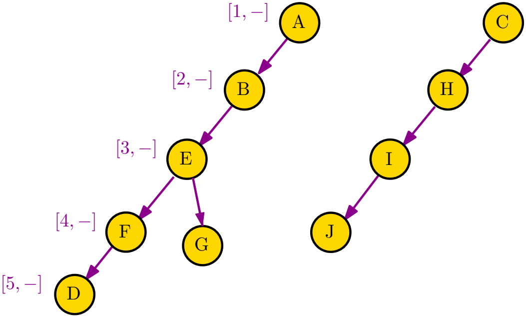

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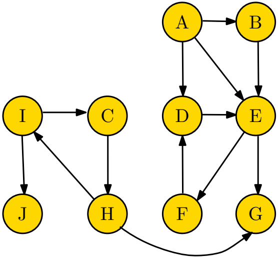

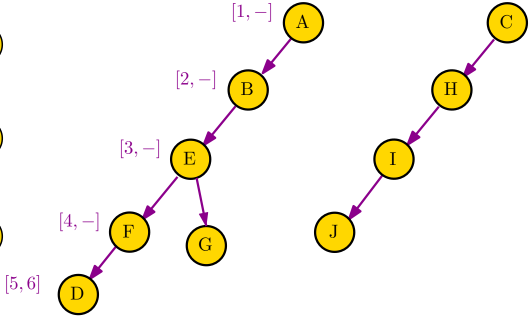

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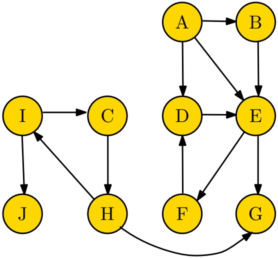

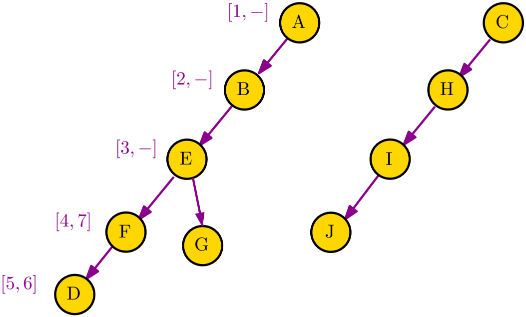

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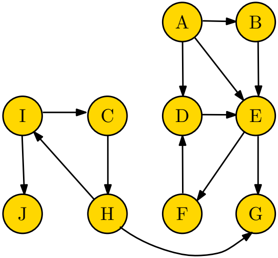

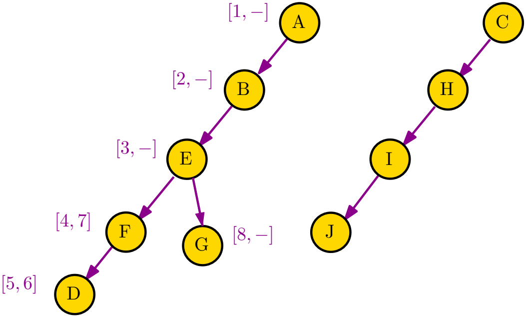

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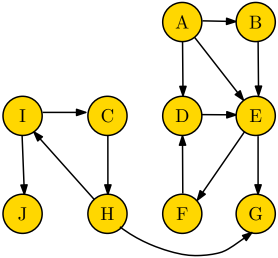

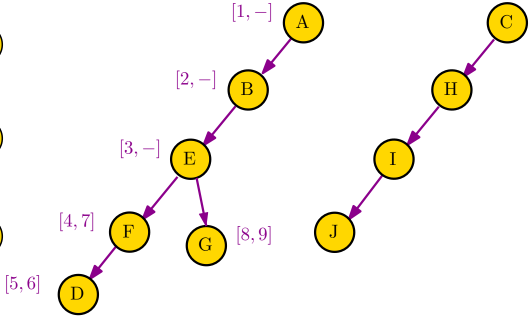

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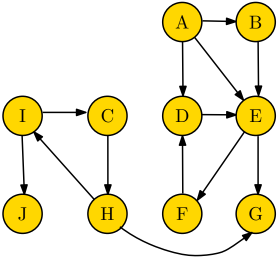

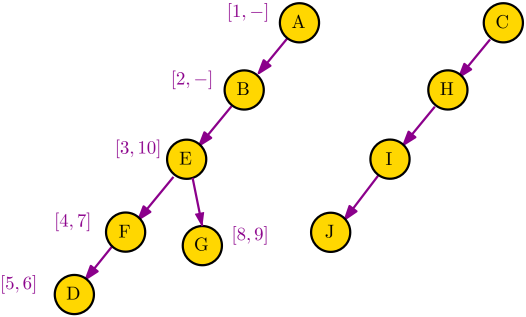

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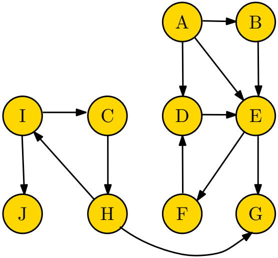

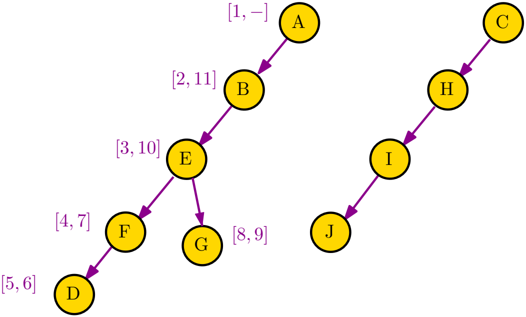

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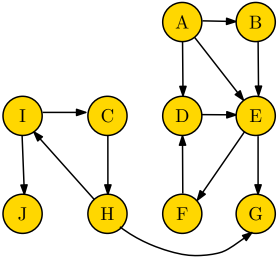

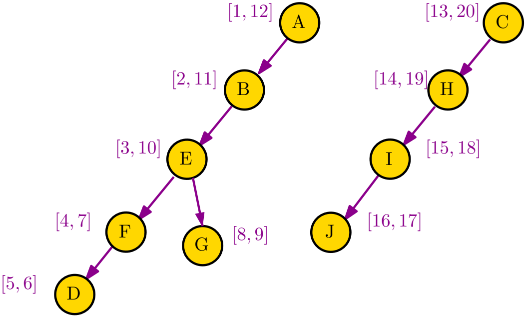

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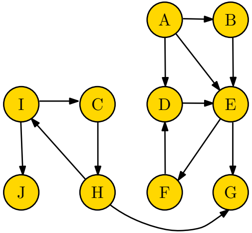

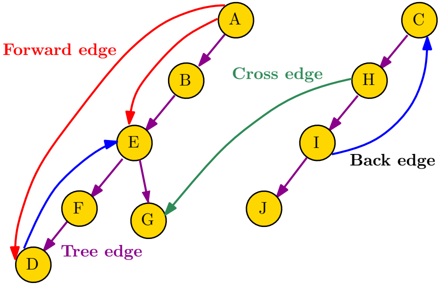

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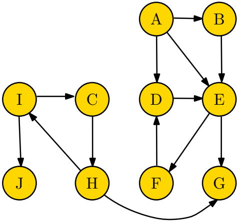
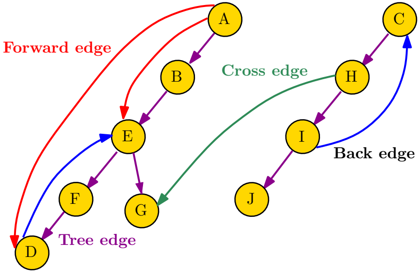
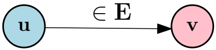
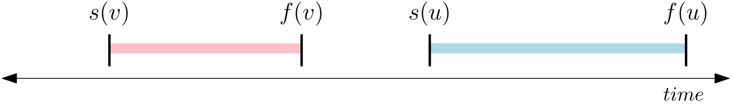
DFS Forest: Identifying Edge Type









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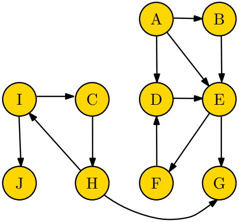
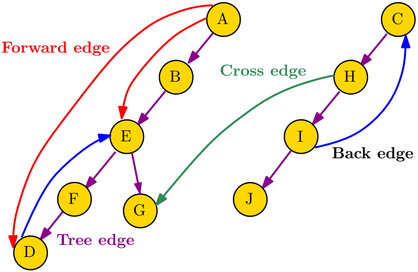
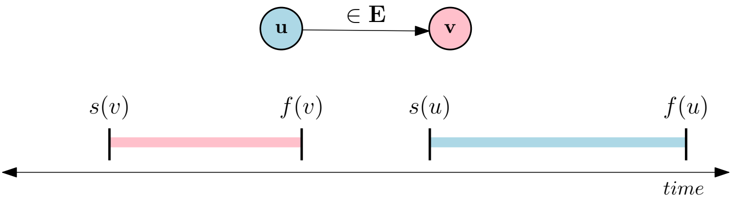
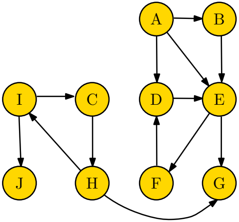


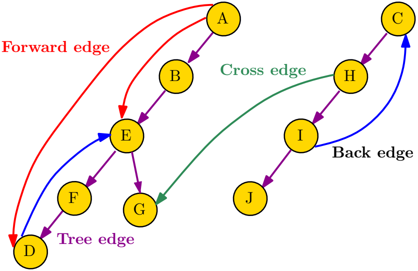


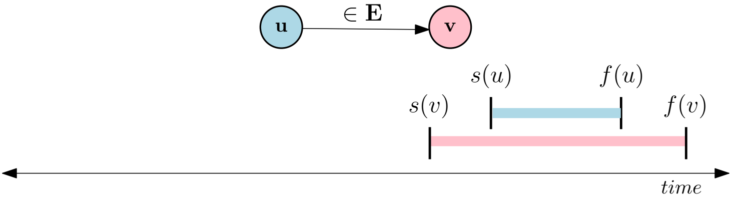


cross edge

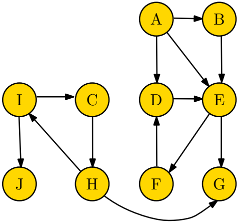
DFS Forest: Identifying Edge Type

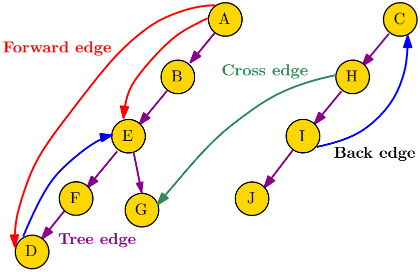


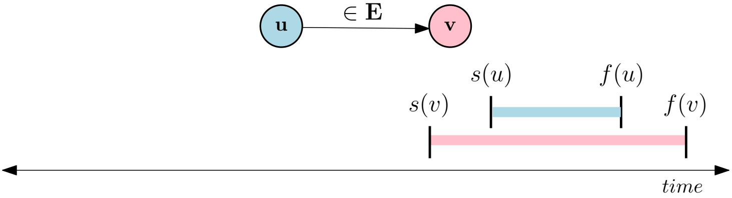




DFS Forest: Identifying Edge Type

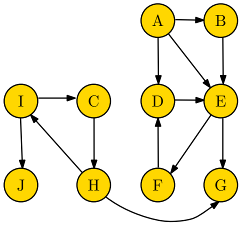


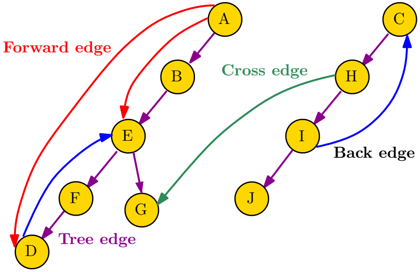


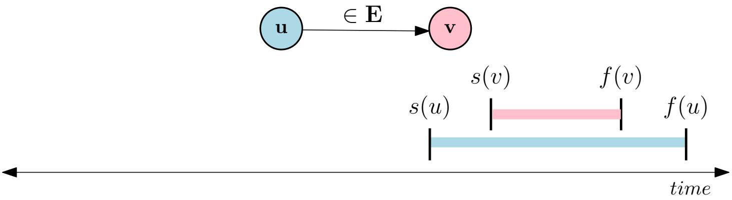


back edge

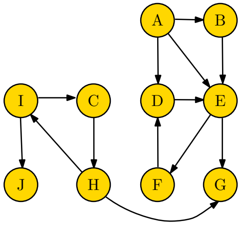
DFS Forest: Identifying Edge Type

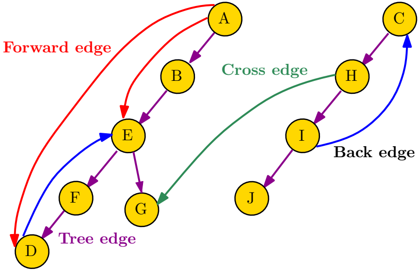


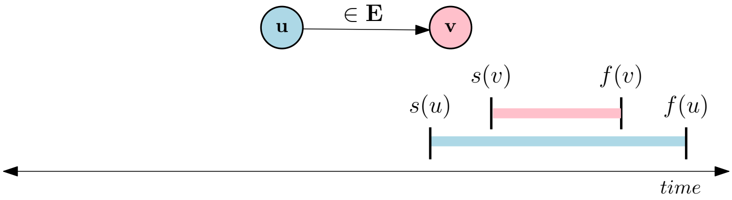




DFS Forest: Identifying Edge Type



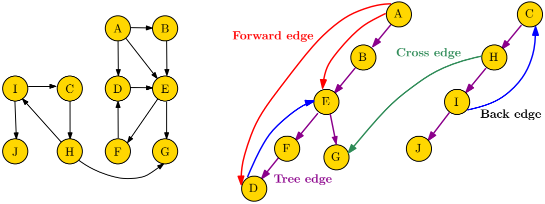


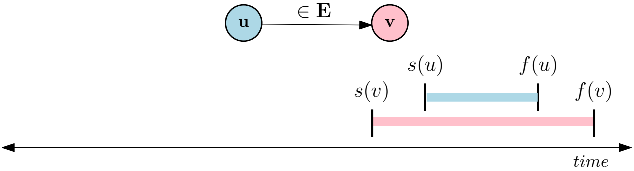


tree or forward edge (check predecessor)

DFS Forest: Cycles in Graphs

Lemma: A digraph G has a directed cycle if and only if the DFS forest of G has a back edge





back edge