Classes of Problems

- Polynomial Time Verification
- The Classes P and NP
- The Classes EXP and coNP
- NP-HARD and NP-COMPLETE Problems
- Proving NP-HARDNESS
- A first NP-COMPLETE Problem

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NP-HARD and NP-COMPLETE Problems

A problem X is NP-HARD, if every problem in NP is polynomial time reducible to X

$$\forall\, Y \in \mathrm{NP}, \quad Y \ \leq_p \ X$$

A problem $X \in \mathrm{NP}$ is NP-COMPLETE, if every problem in NP is polynomial time reducible to X

$$X \in \mathrm{NP} \quad \text{AND} \quad \forall\, Y \in \mathrm{NP}, \quad Y \ \leq_p \ X$$

These problems are at least as hard as any problem in NP

Let NPC be the (sub)class of NP-COMPLETE problems

▷ It is the set of hardest problems in NP

If any NP-complete problem can be solved in poly time, then all problems in NP can be, and thus $\mathrm{P} = \mathrm{NP}$

Proving NP-Complete Problems

A problem X is NP-Complete, if

1. $X \in \mathrm{NP}$
2. $\forall\ Y \in \mathrm{NP}\ Y \leq_p X$

To prove X NP-Complete, reduce an NP-Complete problem Z to X

If Z is NP-Complete, and

1. $X \in \mathrm{NP}$
2. $Z \leq_p X$

then X is NP-Complete

1. $X \in \mathrm{NP}$ is explicitly proved
2. $\forall\ Y \in \mathrm{NP}, \quad Y \leq_p X$ follows by transitivity

   $\forall\ Y \in \mathrm{NP}, \quad Y \leq_p Z$ is true as Z is NP-Complete

   $[Y \leq_p Z \ \wedge\ Z \leq_p X] \implies Y \leq_p X$

A first NP-Complete Problem

To prove X NP-Complete, reduce an NP-Complete problem Z to X

Where to begin? we need a first NP-Complete Problem

Theorem (The Cook-Levin theorem)

SAT(f) *is* NP-Complete

- Proved by Stephen Cook (1971) and earlier by Leonid Levin (but became known later)
- Levin proved six NP-Complete problems (in addition to other results)

We will prove this theorem, but first we prove that the CIRCUIT-SAT(C) problem is NP-Complete and reduce it to the SAT(f) problem

CIRCUIT-SAT(C) is NP-COMPLETE

CIRCUIT-SAT is NP-COMPLETE

First we prove that it is polynomial time verifiable

CIRCUIT-SAT(C) $\in$ NP

The instance C is encoded as a DAG

1. A certificate can be assignment of Boolean values to input wires
2. Verifier finds topological order of C and evaluate output of each gate (node)
3. If the value of sink node is 1, the verifier outputs **Yes**, otherwise **No**

Runtime is clearly polynomial

- Topological sort takes time polynomial in size of input graph
- So is linear scan of vertices to evaluate their value constant time on each

CIRCUIT-SAT(C) is NP-COMPLETE

CIRCUIT-SAT(C) is NP-HARD $\forall X \in \text{NP},\ X \leq_p$ CIRCUIT-SAT(C)

Use the definition of $X \in \text{NP}$ critically

- There is a **poly-sized** certificate S for every instance $\mathcal{I}$ of X and a **poly-time** verifier $\mathcal{V}$ such that $\mathcal{V}(\mathcal{I}, S) =$ **Yes** iff $X(\mathcal{I}) =$ **Yes**
- $\mathcal{I}$ and S have a binary encoding (in digital computers)
- $\mathcal{V}$ can be implemented in a digital computer, takes $\mathcal{I}$ and S and outputs $\mathbf{1}/\mathbf{0}$ in **poly number** of clock cycles
- A computer has a configuration/state (values of all registers (memory), control registers, program counters etc.)
- State changes after each clock cycle according to instruction of $\mathcal{V}$ that are executed by a Boolean combinatorial circuit (the ALU)
- $\mathcal{V}$ outputs $\mathbf{1}/\mathbf{0}$ depending on the final state
- **Ignore the clock and replicate the circuit mapping states to next states**

CIRCUIT-SAT(C) is NP-COMPLETE

CIRCUIT-SAT(C) is NP-HARD $\quad \forall\, X \in \text{NP},\ X \leq_p \text{CIRCUIT-SAT}(C)$

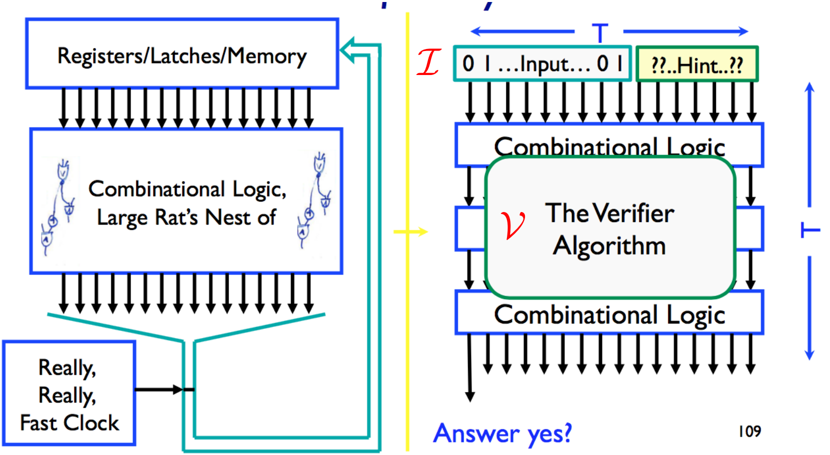

Figure Credit: https://courses.cs.washington.edu/courses/cse421/12wi/

CIRCUIT-SAT(C) is NP-COMPLETE

CIRCUIT-SAT(C) is NP-HARD $\forall X \in \text{NP},\ X \leq_p$ CIRCUIT-SAT(C)

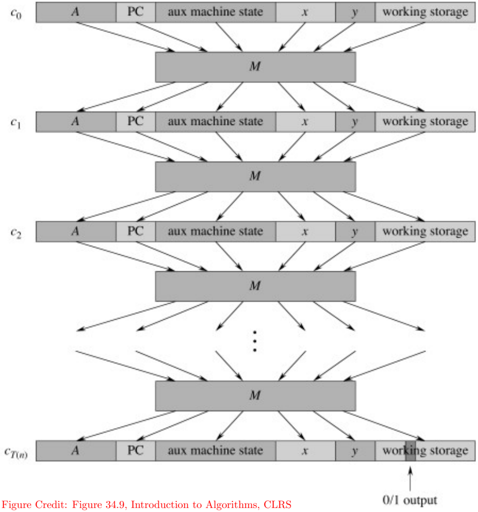

Figure Credit: Figure 34.9, Introduction to Algorithms, CLRS

CIRCUIT-SAT(C) is NP-COMPLETE

CIRCUIT-SAT(C) is NP-HARD $\quad \forall X \in \mathrm{NP},\ X \leq_p$ CIRCUIT-SAT(C)

Let $\mathcal{A}$ be an algorithm to decide the CIRCUIT-SAT(C) problem

We use $\mathcal{A}$ to decide the problem $X \in \mathrm{NP}$ on an instance $\mathcal{I}$

- Construct a circuit C' from the digital implementation of $\mathcal{V}$ on input $\mathcal{I}$
- $\mathcal{A}(C') =$ **Yes** $\iff$ CIRCUIT-SAT$(C') =$ **Yes** means
  - there is an input for which C' outputs **Yes**,
  - meaning there is a certificate S (since $\mathcal{I}$ is hard-coded), on which the verifier $\mathcal{V}$ outputs **Yes**
- Since $\mathcal{V}(\mathcal{I}, S) =$ **Yes** $\iff$ $X(\mathcal{I}) =$ **Yes**, we get an answer for $X(\mathcal{I})$
- Number of stages in C' is polynomial (equal to number of clock cycles, which is polynomial since $\mathcal{V}$ is polynomial time in $|\mathcal{I}|$)
- Number of gates in C' is polynomial, hence constructing it takes poly time