Problem Set: Graphs & Trees

Problem 1. Show that any connected graph on n vertices has at least $n-1$ edges.

Problem 2. Let G be a graph such that the degree of each vertex is ≥ 2. Prove that G has at least one cycle.

Problem 3. Prove that for any graph G, if G is disconnected, then $\overline{G}$ is connected.

Problem 4. Prove that the number of vertices of odd degree in any graph G is even.

Problem 5. Let G be a simple graph. Show that an edge e is a cut edge, if and only if e is not part of any cycle in G.

Problem 6. Prove that if a graph has exactly two vertices of odd degree, then there is a path from one of them to the other.

Problem 7. Let F be a forest with k components. Show that F has $n-k$ edges.

Problem 8. Let G be a connected graph. Prove that in $G-v$ (the graph obtained by removing the vertex v and its incident edges), every connected component has a vertex w_i adjacent to v.

Problem 9. Let G be a simple graph whose vertices all have degree at least k, were $k \geq 1$. Prove:

1. G contains a path of length k.
2. If $k \geq 2$ then G contains a cycle of length at least k.

Problem 10. Show that in every simple graph there is a path from any vertex of odd degree to some other vertex of odd degree.

Problem 11. Show that a simple graph with n vertices and k connected components has at most $\dfrac{(n-k)(n-k+1)}{2}$ edges.

Problem 12. Prove the following theorem:
A full m-ary tree with

1. n vertices has $i = (n-1)/m$ internal vertices and $l = [(m-1)n+1]/m$ leaves
2. i internal vertices has $n = mi+1$ vertices and $l = (m-1)i+1$ leaves
3. l leaves has $n = (ml-1)/(m-1)$ vertices and $i = (l-1)/(m-1)$ internal vertices.

Problem 13. We hinted at the proofs of all of the following statements about tree. In this question you are asked to give their formal but brief proofs. **Note:** *Every tree* $T = (V, E)$, $(|V| = n, |E| = m)$ *satisfies the following properties.*

1. $m = n - 1$
2. There is a unique path between every pair of vertices
3. T is an edge-maximal acyclic graph (i.e. adding any edge to T creates a cycle)
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5. Every edge in T is a cut-edge
6. T is an edge-minimal connected graph (i.e. Removing any edge disconnects T)
7. T has at least two leaves