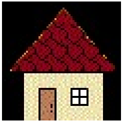
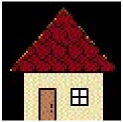
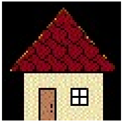
Planar Graphs

- Planar Graphs and Planar Drawings
- Euler's Formula
- Boundaries of Regions
- Degrees of Region
- Face-Edge Handshaking Lemma
- Characterization of Planar Graph

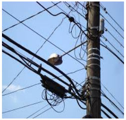
Imdad ullah Khan

Planar Graphs

Is it possible to connect the three houses to the three utilities, such that no connections cross?

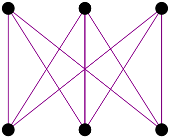




Planar Graphs

Planar Graphs

A graph is planar if it can be drawn in the plane without any edge crossing



$K_{3,3}$

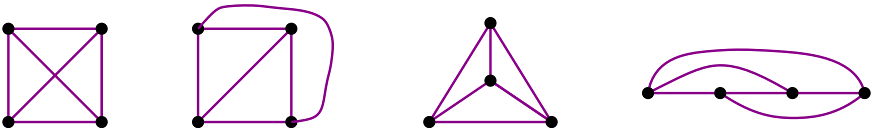
Is it possible to connect the three houses to the three utilities, such that no connections cross?

Is $K_{3,3}$ planar?

Planar Graphs

A graph is planar if it can be drawn in the plane without any edge crossing

Just because in a drawing of G edges are crossing doesn't mean G is not planar



Four different drawings of the same graph, K_4

Planar Graphs

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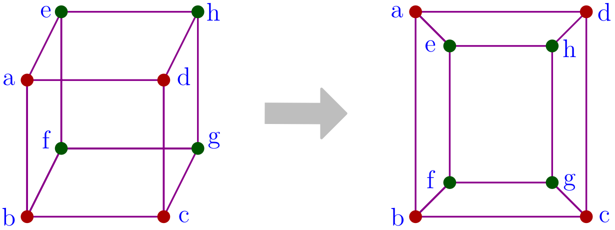
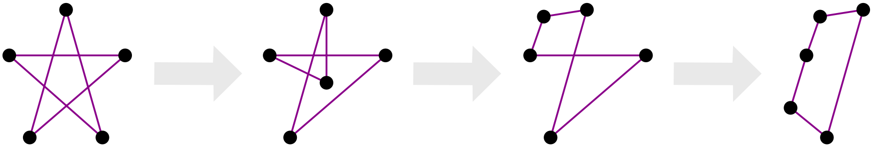


Planar Graphs

A graph is planar if it can be drawn in the plane without any edge crossing

To prove planarity, move the vertices around, redraw the edges without crossing (sometimes in a very indirect faction)



Planar Graphs

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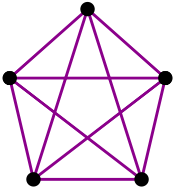
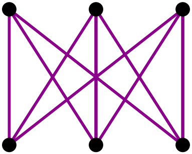
To prove planarity, move the vertices around, redraw the edges without crossing (sometimes in a very indirect faction)

Even harder to prove non-planarity



K_5



$K_{3,3}$

Planar Graphs: A characterization

A graph is planar if it can be drawn in the plane without any edge crossing

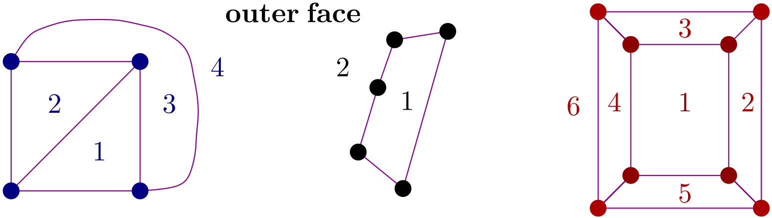
We will find some invariants that all planar graphs satisfy

To prove non-planarity of a graph G, we will show that G doesn't satisfy that invariant

Planar Graphs: A characterization

A plane drawing of a planar graph divides plane into regions or faces, one of them the outer face

A region (or face) is a part of the plane disconnected from other parts by the edges of a graph



Planar Graphs: A characterization

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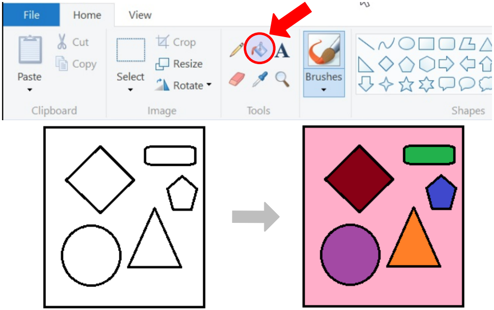
Recall Microsoft Paint **fill** functionality

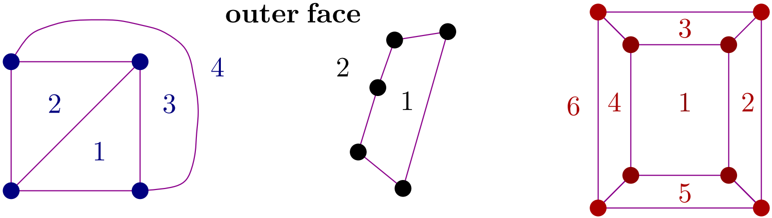


Planar Graphs: A characterization

Euler's Formula: The number of faces of a connected planar graph is invariant of its drawing and is given by

$$f = e - v + 2 \qquad f = |faces|, \;\; e = |E|, \;\; v = |V|$$

Verify it for the following planar graphs



Planar Graphs: Euler Formula

$f = e - v + 2 \qquad f = |faces|,\ \ e = |E|,\ \ v = |V|$

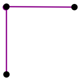
Add edges of G incident to an existing vertex

1) Take one edge of G

$f - e + v = 1 - 1 + 2 = 2$

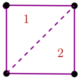
2) Add edge with one existing vertex

$e_{i+1} = e_i + 1,\ \ v_{i+1} = v_i + 1,\ \ f_{i+1} = f_i$



2) Add edge with two existing vertices

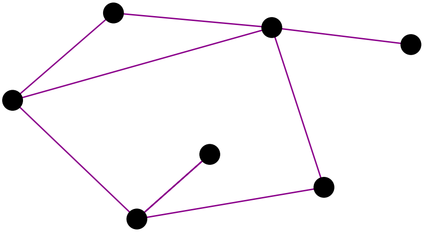
$e_{i+1} = e_i + 1,\ \ v_{i+1} = v_i,\ \ f_{i+1} = f_i + 1$



Planar Graphs: Boundaries of regions

Degree of a region is the number of edges at its boundary

Number of edges encountered in a walk around the boundary of the region



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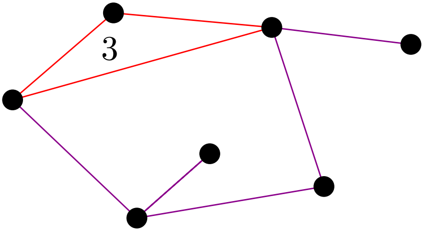

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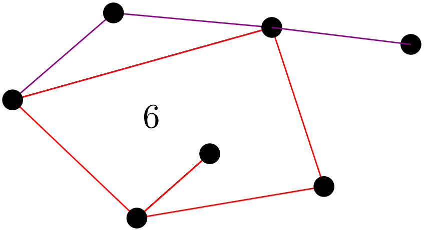

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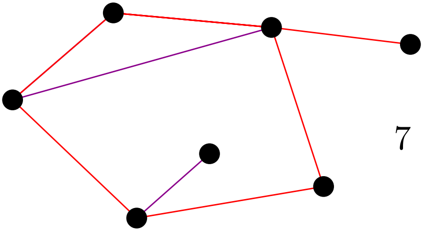

Degree of each region is at least 3

Planar Graphs: Face-Edge Handshaking

Face-Edge Handshaking Lemma

Let G be a planar graph, let R be its regions, then

$$2e = \sum_{F \in R} deg(F)$$

$deg(F) \geq 3 \implies 2e \geq 3f$

From Euler's formula $\quad f - e + v = 2 \implies 2/3e - e + v \geq 2$

Theorem

If G is a connected planar graph with $v \geq 3$, then

$$e \leq 3v - 6$$

Planar Graphs: Face-Edge Handshaking

Theorem

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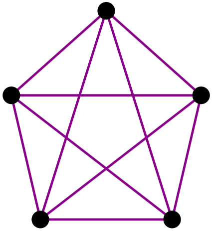
Show that K_5 is not planar



K_5

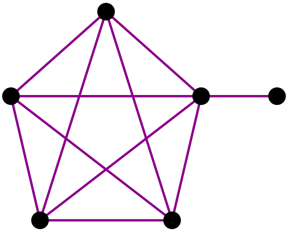
Planar Graphs: Face-Edge Handshaking

Theorem

If G is a connected planar graph with $v \geq 3$, then

$$e \leq 3v - 6$$

The converse is not true! Consider



Planar Graphs: Characterization

Theorem

If G is a connected planar graph with $v \geq 3$, then

$$e \leq 3v - 6$$

Corollary

If G is a connected planar graph, then G has a vertex of degree at most 5

Planar Graphs: Characterization

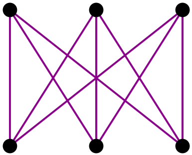
Theorem

If G is a connected planar graph with $v \geq 3$ and no cycles of length 3, then

$$e \leq 2v - 4$$

Same proof as the above, but use the fact that since no cycle is of length 3, thus degree of every region is at least 4

Show that $K_{3,3}$ is not planar



$K_{3,3}$

Planar Graphs: Characterization

Kuratowski's Theorem

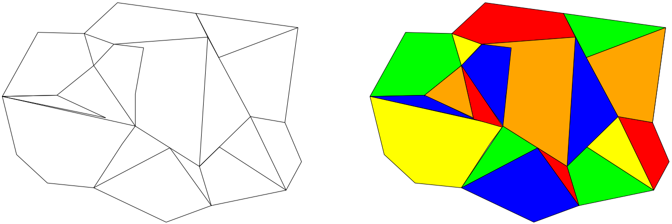
A graph is nonplanar if and only if it contains a subdivision of $K_{3,3}$ or K_5

Coloring Planar Graphs

The PLANAR-GRAPH-COLORING(G) problems: Given a planar graph G, color it with minimum colors

Conjecture (1852)

Regions of any 2-d map can be colored with 4 colors so that no neighboring regions have the same color



Conjecture (1852): Any planar graph can be colored with 4 colors

6-Coloring Planar Graphs

Lemma: Every planar graph has a vertex with degree at most 5

Using this lemma we give a recursive 6-coloring algorithm

▷ Can apply the algorithm to components of disconnected graphs

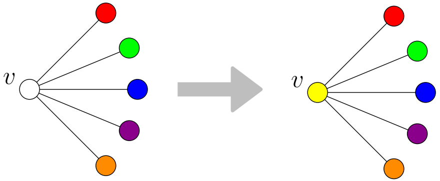
Algorithm 6-COLOR$(G, C = \{c_1, \ldots, c_6\})$

Let $v \in V$ such that $deg(v) \leq 5$

6-COLOR$(G - v, C = \{c_1, \ldots, c_6\})$

Let $C' \subset C$ be the set of colors used for $N(v)$ ▷ $|C'| \leq 5$

Color v with a color in $C \setminus C'$



Graphs Applications: Coloring

- Kempe (1879) announced a proof
- Tait (1880) announced an alternative proof
- Heawood (1890) found a flaw in Kempe's proof
- Petersen (1881) found a flaw in Tait's proof
- Heesch (1969) reduced the statement to checking a large number of cases
- Appel & Haken (1976) gave a "proof", that involved a computer program to check 1936 cases (1200 hours of computer time)
- Robertson et.al. (1997) gave another simpler "proof" but still involved computer program

UIUC stamp in honor of the 4-Color theorem

- No human can check all the cases
- What if the program has a bug
- What if the compiler/system hardware has a bug