Graphs

- Graphs are everywhere
- Types and Terminology: Handshaking lemma
- Representation, Complement, Transpose, Subgraph
- Walks, Paths and Cycles
- (Strongly) Connected and k-Connected graphs
- Applications: BFS, DFS, Eulerian graphs
- Advanced Applications: Optimization & Massive Graph Analysis

Imdad ullah Khan

Graphs

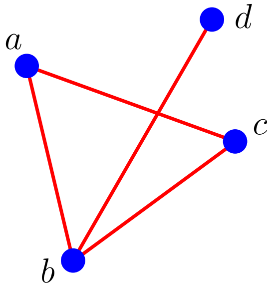
A graph is

1. a set of vertices V
2. a set of edges E; each edge is a 2-subset of V

$$G = \big(V, E\big)$$

$$V = \big\{a, b, c, d\big\}$$

$$E = \big\{(a, b), (a, c), (b, c), (b, d)\big\}$$



Each edge is an unordered pair from V, we use the ordered pair notation

Types of Graphs: Undirected Simple Graphs

Applications require different types of graphs

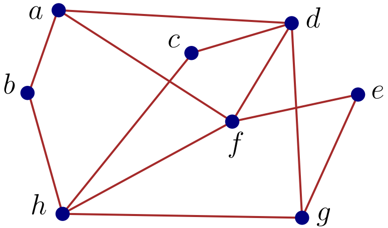
Undirected Simple Graphs

$G = (V, E)$

V is set of vertices

E is set of edges

(2-subsets of V, unordered pairs)



Types of Graphs: Undirected Simple Graphs

Let $G = (V, E)$ be a simple graph

Order of the graph is number of vertices in it, $|V| = n$

Size of the graph is number of edges in it, $|E|$

$\boxed{\textbf{ICP 14-01}}$ What is the maximum possible size of G of order n?

$$\binom{n}{2}$$

Types of Graphs: PseudoGraphs and Multigraphs

Applications require different types of graphs

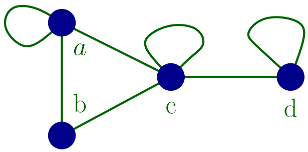
PseudoGraphs

$G = (V, E)$

V is set of vertices

E is set of edges

(self loops allowed)



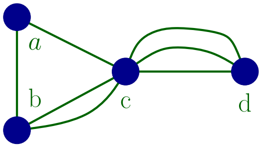
Multigraphs

$G = (V, E)$

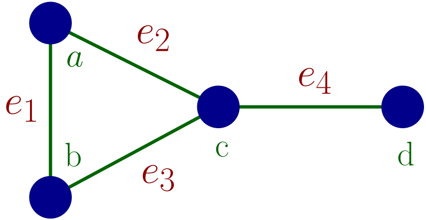
V is set of vertices

E is multi-set of edges

may have self loops too

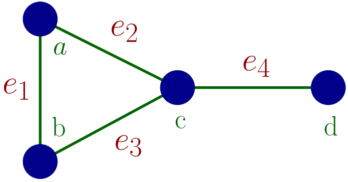


Graph Terminology: Incidence



- $e_1 = (a, b)$: a and b are endpoints of e_1 ▷ slight abuse of notation
- a and b are **adjacent**
- e_1 is incident to a and b

Graph Terminology: Degree



Degree of a vertex is the number of edges incident on it

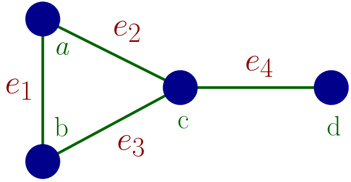
The number of vertices adjacent to a vertex

Denoted by $deg(v)$

ICP 14-02 What is the min and max possible degree of a vertex?

$$0 \leq deg(v) \leq n-1$$

Graph Terminology: Degree



In this graph

$\boxed{\textbf{ICP 14-03}}$ $deg(a) = ?$

$\boxed{\textbf{ICP 14-04}}$ $deg(d) = ?$

$\boxed{\textbf{ICP 14-05}}$ $deg(b) = ?$

$\boxed{\textbf{ICP 14-06}}$ $deg(c) = ?$

Types of Graphs: Directed Graphs (digraphs)

Applications require different types of graphs

Directed Graphs (digraphs)

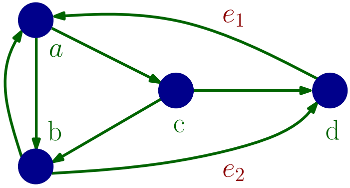
$G = (V, E)$

V is set of vertices

E is set of edges (ordered pairs)

$E \subseteq V \times V$ (ordered pairs)

irreflexive relation on V



Types of Graphs: Directed Graphs (digraphs)

Let $G = (V, E)$ be a simple digraph

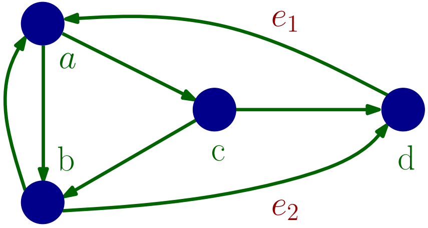
Order of the graph is number of vertices in it, $|V| = n$

Size of the graph is number of edges in it, $|E|$

$\boxed{\textbf{ICP 14-07}}$ What is the maximum possible size of G of order n?

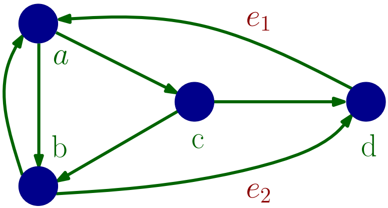
$$n(n-1)$$

Graph Terminology: Directed Edges



- $e_1 = (d, a) \neq (a, d)$
- d is source of e_1 and a is target of e_1

Graph Terminology: In-degree and Out-degree



in-degree of a vertex is the number of (directed) edges incoming into it

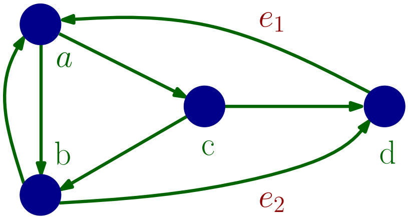
The number of edges with this vertex as a target

Denoted by $deg^-(v)$ or $d^-(v)$

ICP 14-08 What is the min and max possible in-degree of a vertex?

$$0 \leq deg^-(v) \leq n-1$$

Graph Terminology: In-degree and Out-degree



out-degree of a vertex is the number of (directed) edges outgoing form it

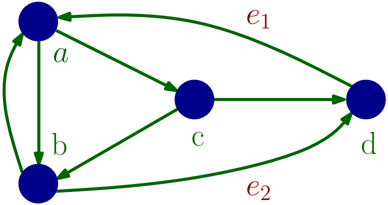
The number of edges with this vertex as a source

Denoted by $deg^+(v)$ or $d^+(v)$

ICP 14-09 What is the min and max possible out-degree of a vertex?

$$0 \leq deg^+(v) \leq n-1$$

Graph Terminology: In-degree and Out-degree



In this graph

ICP 14-10	$deg^+(a) \; = ?$	**ICP 14-14**	$deg^-(a) \; = ?$
ICP 14-11	$deg^+(d) \; = ?$	**ICP 14-15**	$deg^-(d) \; = ?$
ICP 14-12	$deg^+(b) \; = ?$	**ICP 14-16**	$deg^-(b) \; = ?$
ICP 14-13	$deg^+(c) \; = ?$	**ICP 14-17**	$deg^-(c) \; = ?$

Pigeonhole Principle

- Party of n people
- Some handshaking takes place
- A pair either shake hands once or do not

At least 2 people shook same number of hands

Pigeons: The $n \geq 2$ people

Pigeon Holes: Possible number of handshakes

$$\{0, 1, 2, \ldots, n-1\}$$

Pigeonhole Principle

At least two people shook the same number of hands

Pigeons: The $n \geq 2$ people

Pigeon Holes: Possible number of handshakes

$$\{0, 1, 2, \ldots, n-1\}$$

$$|\text{pigeon holes}| = |\text{pigeons}|$$

Case 1: Someone shook $n-1$ hands

No one has 0 handshakes. n pigeons, $n-1$ pigeon holes

Case 2: No one shook $n-1$ hands

No one has $n-1$ handshakes. n pigeons, $n-1$ pigeon holes

Graph Terminology: Degree

Theorem

In any graph G there are at least 2 vertices of the same degree

ICP 14-18 Give a formal proof of this theorem

Pigeons: The $n \geq 2$ vertices

Pigeon Holes: Possible values of degrees $\{0, 1, 2, \ldots, n-1\}$

Case 1: Some vertex has degree $n-1$

▷ No vertex degree 0. n pigeons, $n-1$ pigeon holes

Case 2: No vertex has degree $n-1$

▷ No vertex has degree $n-1$. n pigeons, $n-1$ pigeon holes

Hey! Graphs also model Counting Problems

Proof using Case Analysis

Given any two persons they are either friends or they are strangers

Theorem

Every collection of 6 *people contains a group of* 3 *friends or a group of* 3 *strangers*

Let $\{a, b, c, d, e, f\}$ be an arbitrary collection of 6 people

Consider a (fixed) person a

Case 1: Among the remaining there are ≥ 3 people who are all friends with a

Case 2: Among the remaining there are ≥ 3 people who are all strangers to a

One of these two cases must happen

Proof using Case Analysis

Theorem

Every collection of 6 people contains a group of 3 friends or a group of 3 strangers

Case 1: Among the remaining there are ≥ 3 people who are all friends with a

Case 1.1:
Among the ≥ 3 friends of a there are two who are friends with each other

Case 1.2:
All the ≥ 3 friends of a are strangers to each other

All subcases covered

Proof using Case Analysis

Theorem

Every collection of 6 *people contains a group of* 3 *friends or a group of* 3 *strangers*

Case 2: Among the remaining there are ≥ 3 people who are all strangers to a

Case 2.1:
Among the ≥ 3 strangers to a there are two who are strangers to each other

Case 2.2: All the ≥ 3 strangers to a are friends with each other

All subcases covered

Proof by Case Analysis

Theorem

Any Graph on 6 *vertices has* 3 *vertices all adjacent to each other or* 3 *vertices all non-adjacent to each other*

Proof is exactly the same as above - model pairwise friendships with edges and being strangers with lack of edges

Practice with degree

On a piece of paper draw the following very simple graphs

Make a graph on 3 vertices with degrees 2, 2, and 1

There are 5 people and every one is friend with exactly three other people. Model this with graph, vertices are people, edges are friendships

Do not spend more than 60 seconds on either problem

Ok, you can spend the next 600 years on them!

Handshaking Lemma: Undirected Graphs

Make a graph on 3 vertices with degrees 2, 2, and 1

There are 5 people and every one is friend with exactly three other people. Model this with graph, vertices are people, edges are friendships

Such graphs do not exist!

Handshaking Lemma: Undirected Graphs

Theorem (The Handshaking Lemma)

The sum of the degrees of vertices in a graph is even

Theorem (The Handshaking Lemma (A more precise version))

$$\sum_{v} deg(v) = 2|E|$$

Theorem (The Handshaking Lemma (Another version))

The number of odd degree vertices in a graph is even

ICP 14-19 Argue that these three statements are equivalent.

ICP 14-20 Give a formal proof of the handshaking lemma.

Handshaking Lemma: Directed Graphs

Theorem (The Handshaking Lemma (Diagraph version))

$$\sum_{v} deg^{+}(v) = \sum_{v} deg^{-}(v) = |E|$$

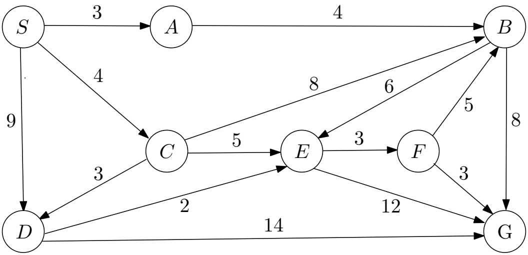
ICP 14-21 Give a formal proof of the digraph's handshaking lemma.

Types of Graphs: Weighted Graphs (digraphs)

Some applications work on graphs with weights on edges

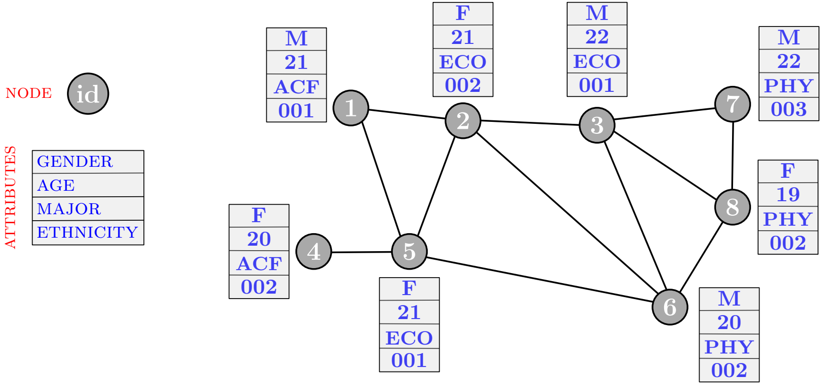
Weighted Graphs (digraphs) : $G = (V, E, w)$

- V : Set of vertices
- E : Set of edges (directed edges)
- w : cost/weight function: $w : E \to \mathbb{R}$
  - ▷ weights could be lengths, airfare, toll, energy



Types of Graphs: Attributed Graphs

- Each element (vertex/edge) has associated properties
- It can be directed/undirected



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