Recursive Definition and Recurrence Relations

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Solution of Recurrence Relation

Recurrence relation and initial conditions uniquely determine a sequence

A sequence is called a **solution** of a recurrence relation if its terms satisfy the recurrence relation

A closed form of a recurrence is a formula that on input n returns nth value of the sequence

- ▷ Called **closed form** as it does not use previous terms in evaluation

Solving a recurrence relation means finding a closed form for it

Evaluating Recursion without closed form

Recurrence relation for sequence of factorials $\{a_n\}$

$$a_n = \begin{cases} na_{n-1} & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{cases}$$

For a fixed n, $a_n = n!$ can be evaluated as follows

Recurrence Relation: $a_n = na_{n-1}$

a_4: $4! = 4(3!)$ .... $= 4(6) = 24$

a_3: $3! = 3(2!)$ .... $= 3(2) = 6$

a_2: $2! = 2(1!)$ .... $= 2(1) = 2$

a_1: $1! = 1(0!)$ ..... $= 1(1) = 1$

a_0: Initial Term: $0! = 1$

Number of arithmetic operations (multiplications) to evaluate this first order recurrence relation to compute the nth term is n

Solving Recurrence relation

We discuss the following two method to solve a recurrence

1. Make a calculated guess for a closed from and prove it by induction
   - A reasonably good closed form can usually be guessed by observing the pattern in the first few terms
   - Can try to disprove a guessed closed form by a single counter example
2. Use the substitution method

Proving Recurrence Closed Form by Induction

$$H_n = \begin{cases} 2H_{n-1} + 1 & \text{if } n > 1 \\ 1 & \text{if } n = 1 \end{cases}$$

Make a calculated guess for a closed from and prove it by induction

Guess: $H_n = 2^n - 1$

Proof by induction on n:

Basis Step: $P(1) : H_1 = 2^1 - 1 = 1$ ▷ hence, true

Inductive Hypothesis: $P(k-1)$ is true i.e. $H_{k-1} = 2^{k-1} - 1$

Inductive Step: Prove $P(k-1) \rightarrow P(k)$ is true

$$H_k = 2(H_{k-1}) + 1 = 2(2^{k-1} - 1) + 1 = 2^k - 1$$

Proving Recurrence Closed Form by Induction

$$r_n = \begin{cases} 2r_{n-1} - r_{n-2} & \text{if } n \geq 2 \\ 3 & \text{if } n = 1 \\ 0 & \text{if } n = 0 \end{cases}$$

Make a calculated guess for a closed from and prove it by induction

Guess: $r_n = 3n$

ICP 13-8

- Check a few initial terms
- If they are correct, prove by induction

Proving Recurrence Closed Form by Induction

$$r_n = \begin{cases} 2r_{n-1} - r_{n-2} & \text{if } n \geq 2 \\ 5 & \text{if } n = 1 \\ 5 & \text{if } n = 0 \end{cases}$$

Make a calculated guess for a closed from and prove it by induction

Guess: $r_n = 5$

ICP 13-9

- Check a few initial terms
- If they are correct, prove by induction

Note that the recurrence is the same, sequence is different just because initial conditions are different

Proving Recurrence Closed Form by Induction

$$r_n = \begin{cases} 2r_{n-1} - r_{n-2} & \text{if } n \geq 2 \\ 2 & \text{if } n = 1 \\ 1 & \text{if } n = 0 \end{cases}$$

Make a calculated guess for a closed from and prove it by induction

Guess: $r_n = 2^n$

ICP 13-10

- Check a few initial terms
- If they are correct, prove by induction

- $r_0 = 1 = 2^0$ $\quad r_1 = 2 = 2^1$ $\quad r_2 = 2(2) - 1 = 3 \neq 4 = 2^2$

Proving Recurrence Closed Form by Induction

$\{t_n\} = 0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, 120, 136$

Make a calculated guess for a closed from and prove it by induction

$$t_n = \begin{cases} 0 & \text{if } n = 0 \\ t_{n-1} + n & \text{if } n \geq 1 \end{cases}$$

Triangular Numbers

Guess: $$t_n = \frac{n(n+1)}{2}$$

Proof by induction on n:

Basis Step: $P(0) : t_0 = {}^{0(0+1)}/_2 = 0$ ▷ hence, true

Inductive Hypothesis: $P(k-1)$ is true i.e. $t_{k-1} = {}^{(k-1)(k)}/_2$

Inductive Step: Prove $P(k-1) \rightarrow P(k)$ is true

$$t_k = t_{k-1} + k = {}^{(k-1)(k)}/_2 + k = {}^{k(k+1)}/_2$$

Solving Recurrences using Substitution

$$H_n = \begin{cases} 2H_{n-1} + 1 & \text{if } n > 1 \\ 1 & \text{if } n = 1 \end{cases}$$

Substitute the value of function until a pattern becomes apparent

$$\begin{aligned} H_n &= 2H_{n-1} + 1 \\ &= 2(2H_{n-2} + 1) + 1 \quad = \quad 2^2 H_{n-2} + 2^1 + 1 \\ &= 2^2(2H_{n-3} + 1) + 2 + 1 \quad = \quad 2^3 H_{n-3} + 2^2 + 2^1 + 1 \\ &= 2^3(2H_{n-4} + 1) + 2^2 + 2 + 1 \quad = \quad 2^4 H_{n-4} + 2^3 + 2^2 + 2^1 + 1 \\ &\;\vdots \\ &= 2^{n-1} H_1 + 2^{n-2} + 2^{n-3} + \cdots + 2 + 1 \\ &= 2^{n-1} + 2^{n-2} + \cdots + 2 + 1 \quad = \quad 2^n - 1 \end{aligned}$$

Solving Recurrences using Substitution

Substitute the value of function until a pattern becomes apparent

$$T(n) = \begin{cases} T(\frac{n}{2}) + 5 & \text{if } n > 1 \\ 5 & \text{if } n = 1 \end{cases}$$

$$\begin{aligned}
T(n) &= T(n/2) + 5 \\
&= (T(n/4) + 5) + 5 \\
&= (T(n/8) + 5) + 5 + 5 \\
&= (T(n/16) + 5) + 5 + 5 + 5 \\
&\vdots \\
&= T(n/2^k) + \underbrace{5 + 5 \ldots + 5}_{k \text{ times}} \\
&\vdots \\
&= T(n/2^{\log n}) + \underbrace{5 + 5 + \ldots. + 5}_{\text{log n}} \\
&= T(1) + 5 \log n \\
&= 5 + 5 \log n \;=\; 5(1 + \log n)
\end{aligned}$$

Solving Recurrences using Substitution

$$T(n) = \begin{cases} 2T(\frac{n}{2}) + n & n > 1 \\ 1 & \text{if } n = 1 \end{cases}$$

$$\begin{aligned}
T(n) &= 2T(n/2) + n \\
&= 2(2T(n/4) + n/2) + n = 2*2T(n/4) + n + n \\
&= 2*2*2T(n/8) + n + n + n \\
&= 2*2*2*2T(n/16) + n + n + n + n \\
&\vdots \qquad \vdots \\
&= \underbrace{2*2*2...*2}_{\text{k}} * T(n/2^k) + \underbrace{n + n + ... + n}_{\text{k}} \\
&\vdots \qquad \vdots \\
&= \underbrace{2*2*2...*2}_{\text{log n}} * T(n/2^{\log n}) + \underbrace{n + n + ... + n}_{\text{log n}} \\
&= 2^{\log n}T(1) + n\log n \;=\; n + n\log n
\end{aligned}$$

Solving Recurrences using Substitution

$$T(n) = \begin{cases} 4T(n/2) + 3n & n > 1 \\ 1 & n = 1 \end{cases}$$

$$T(n) = 4T(n/2) + 3n \;=\; 4(4T(n/4) + 3n/2) + 3n$$

$$= 4(4(4T(n/8) + 3n/4) + 3n/2) + 3n = 4*4*4T(n/8) + 4*4 3n/4 + 4 3n/2 + 3n$$

$$\vdots \qquad \vdots$$

$$= \underbrace{4*4*4\ldots*4}_{\text{k}} * T(n/2^k) + \sum_{i=0}^{k-1} (4^i * 3n/2^i)$$

$$\vdots \qquad \vdots$$

$$= \underbrace{4*4*4\ldots*4}_{\log n} * T(n/2^{\log n}) + \sum_{i=0}^{\log n - 1} 2^i * 3n \quad = \quad 4^{\log n} * 1 + 3n(2^{\log n})$$

$$= 2^{2\log n} + 3n*n \;=\; n^2 + 3n^2 \;=\; 4n^2$$