Binomial Theorem and Pascal Triangle

- The Binomial Theorem
- The Pascal Triangle (or the al-Karaji triangle)
- Patterns in the Pascal Triangle (mod n)

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The Binomial Theorem

We know these "formulae"

$$(1+x)^0 = 1$$
$$(1+x)^1 = 1 + 1x$$
$$(1+x)^2 = 1 + 2x + 1x^2$$
$$(1+x)^3 = 1 + 3x + 3x^2 + 1x^3$$
$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + 1x^4$$
$$(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + 1x^5$$
$$(1+x)^6 = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + 1x^6$$

Focus on the coefficients in these expressions

The Pascal Triangle

$$
\begin{array}{ccccccccccc}
 & & & & & \mathbf{1} & & & & & \\
 & & & & \mathbf{1} & & \mathbf{1} & & & & \\
 & & & \mathbf{1} & & \mathbf{2} & & \mathbf{1} & & & \\
 & & \mathbf{1} & & \mathbf{3} & & \mathbf{3} & & \mathbf{1} & & \\
 & \mathbf{1} & & \mathbf{4} & & \mathbf{6} & & \mathbf{4} & & \mathbf{1} & \\
\mathbf{1} & & \mathbf{5} & & \mathbf{10} & & \mathbf{10} & & \mathbf{5} & & \mathbf{1} \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
$$

The Pascal Triangle

$$
\begin{array}{ccccccccccccccccc}
 & & & & & & & & \mathbf{1} & & & & & & & & \\
 & & & & & & & \mathbf{1} & & \mathbf{1} & & & & & & & \\
 & & & & & & \mathbf{1} & & \mathbf{2} & & \mathbf{1} & & & & & & \\
 & & & & & \mathbf{1} & & \mathbf{3} & & \mathbf{3} & & \mathbf{1} & & & & & \\
 & & & & \mathbf{1} & & \mathbf{4} & & \mathbf{6} & & \mathbf{4} & & \mathbf{1} & & & & \\
 & & & \mathbf{1} & & \mathbf{5} & & \mathbf{10} & & \mathbf{10} & & \mathbf{5} & & \mathbf{1} & & & \\
 & & \mathbf{1} & & \mathbf{6} & & \mathbf{15} & & \mathbf{20} & & \mathbf{15} & & \mathbf{6} & & \mathbf{1} & & \\
 & \mathbf{1} & & \mathbf{7} & & \mathbf{21} & & \mathbf{35} & & \mathbf{35} & & \mathbf{21} & & \mathbf{7} & & \mathbf{1} & \\
\mathbf{1} & & \mathbf{8} & & \mathbf{28} & & \mathbf{56} & & \mathbf{70} & & \mathbf{56} & & \mathbf{28} & & \mathbf{8} & & \mathbf{1} \\
\binom{8}{0} & & \binom{8}{1} & & \binom{8}{2} & & \binom{8}{3} & & \binom{8}{4} & & \binom{8}{5} & & \binom{8}{6} & & \binom{8}{7} & & \binom{8}{8}
\end{array}
$$

The Pascal Triangle

Observe how the next row is obtained from the previous

Pascal's Triangle

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pascal Identity

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

$\binom{8}{0}$ $\binom{8}{1}$ $\binom{8}{2}$ $\binom{8}{3}$ $\binom{8}{4}$ $\binom{8}{5}$ $\binom{8}{6}$ $\binom{8}{7}$ $\binom{8}{8}$

The Pascal Triangle

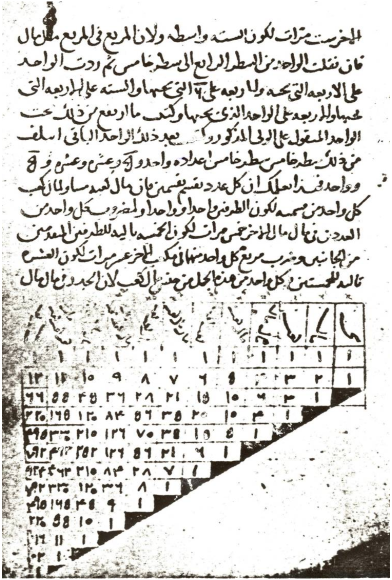
Abu Bakr b. Muhammad b. Husayn al-Karaji

or al-Karakhi

(953 - 1029)

Blaise Pascal

(1623 - 1662)



$$\binom{n}{k} = \binom{n}{n-k}$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

The Pascal Triangle

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\mathbf{1} = 1$$

$$\mathbf{1} + \mathbf{1} = 2$$

$$\mathbf{1} + \mathbf{2} + \mathbf{1} = 4$$

$$\mathbf{1} + \mathbf{3} + \mathbf{3} + \mathbf{1} = 8$$

$$\mathbf{1} + \mathbf{4} + \mathbf{6} + \mathbf{4} + \mathbf{1} = 16$$

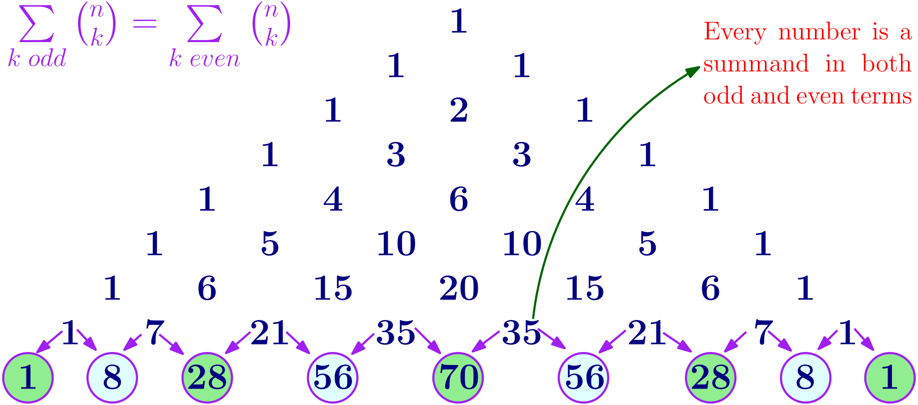
$$\mathbf{1} + \mathbf{5} + \mathbf{10} + \mathbf{10} + \mathbf{5} + \mathbf{1} = 32$$

$$\mathbf{1} \quad \mathbf{6} \quad \mathbf{15} \quad \mathbf{20} \quad \mathbf{15} \quad \mathbf{6} \quad \mathbf{1}$$

$$\mathbf{1} \quad \mathbf{7} \quad \mathbf{21} \quad \mathbf{35} \quad \mathbf{35} \quad \mathbf{21} \quad \mathbf{7} \quad \mathbf{1}$$

$$\mathbf{1} \quad \mathbf{8} \quad \mathbf{28} \quad \mathbf{56} \quad \mathbf{70} \quad \mathbf{56} \quad \mathbf{28} \quad \mathbf{8} \quad \mathbf{1}$$

$$\sum_{k\ odd} \binom{n}{k} = \sum_{k\ even} \binom{n}{k}$$



Every number is a summand in both odd and even terms

The Pascal Triangle

$$\sum_{j=r}^{n} \binom{j}{r} = \binom{n+1}{r+1}$$

[Rosen] Thm 4, pp 368

1								
1	1							
1	2	1						
1	3	3	1					
1	4	6	4	1				
1	5	10	10	5	1			
1	6	15	20	15	6	1		
1	7	21	35	35	21	7	1	
1	8	28	56	70	56	28	8	1

The Pascal Triangle

$$\sum_{j=r}^{n} \binom{j}{r} = \binom{n+1}{r+1}$$

[Rosen] Thm 4, pp 368

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

The Pascal Triangle

$$\sum_{j=r}^{n} \binom{j}{r} = \binom{n+1}{r+1}$$

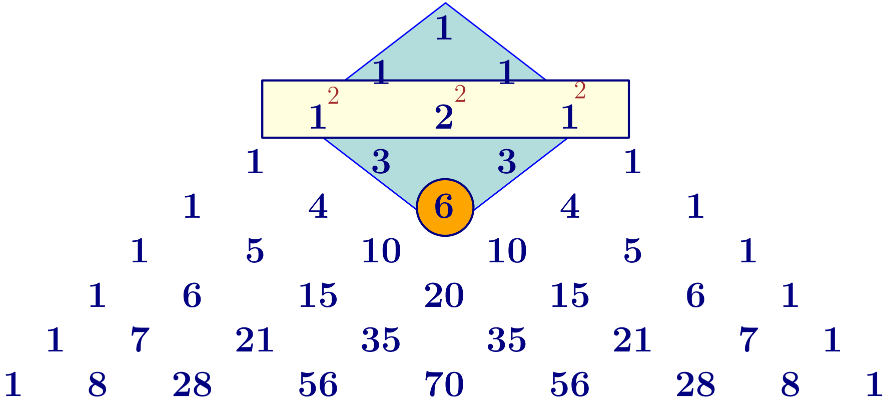
[Rosen] Thm 4, pp 368

								1								
							1		**1**							
						1		**2**		**1**						
					1		**3**		**3**		**1**					
				1		**4**		**6**		**4**		**1**				
			1		**5**		**10**		**10**		**5**		**1**			
		1		**6**		**15**		**20**		**15**		**6**		**1**		
	1		**7**		**21**		**35**		**35**		**21**		**7**		**1**	
1		**8**		**28**		**56**		**70**		**56**		**28**		**8**		**1**

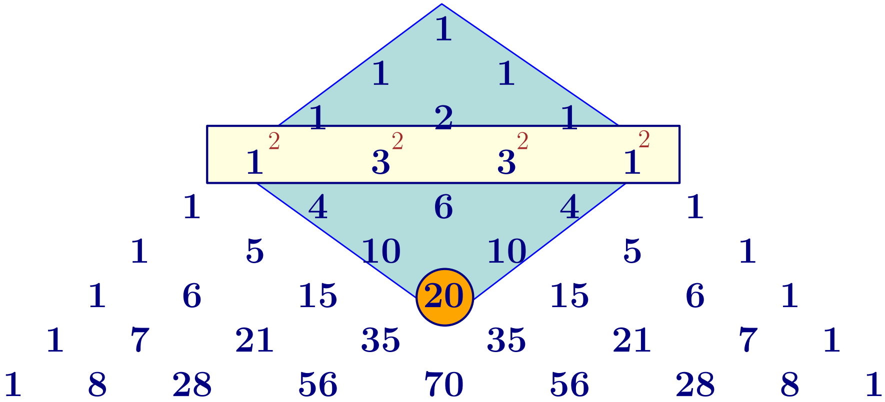
The Pascal Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

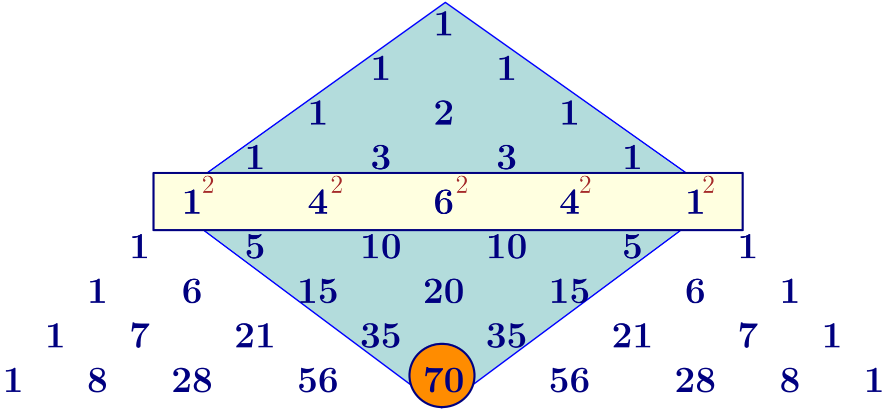
The Pascal Triangle



The Pascal Triangle



The Pascal Triangle



The Pascal Triangle

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Corollary 4, pp 368