Binomial Theorem and Pascal Triangle

- The Binomial Theorem
- The Pascal Triangle
- Patterns in the Pascal Triangle (mod n)

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The Pascal Identity

The Pascal Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Combinatorial Proof:

Counting Task: Select k-subset of an n-set

- LHS counts the number of k-subsets of an n-set
- Fix an element x in the n-set. RHS counts two types of k-subsets.

Those $\underbrace{\text{containing the element } x}_{\binom{n-1}{k-1}}$ and $\underbrace{\text{those not containing } x}_{\binom{n-1}{k}}$

The Binomial Theorem

We know these "formulae"

$$
\begin{aligned}
(1+x)^0 &= 1 \\
(1+x)^1 &= 1 + 1x \\
(1+x)^2 &= 1 + 2x + 1x^2 \\
(1+x)^3 &= 1 + 3x + 3x^2 + 1x^3 \\
(1+x)^4 &= 1 + 4x + 6x^2 + 4x^3 + 1x^4 \\
(1+x)^5 &= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + 1x^5 \\
(1+x)^6 &= 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + 1x^6
\end{aligned}
$$

Focus on the coefficients in these expressions

The Binomial Theorem

$$(x+y)^2 \;=\; x^2 + 2xy + y^2$$

$$(x+y)^2 \;=\; (x+y)(x+y)$$

Make all possible products of one term each in the first and second factor

Each term has form $x^k y^{2-k}$

▷ total degree of a term in $(x+y)^2$ is 2

$$(x+y)^2 \quad = \quad x\cdot x \;+\; x\cdot y \;+\; y\cdot x \;+\; y\cdot y$$

Sum the identical terms

The Binomial Theorem

$$(x+y)^3 \; = \; x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^3 \; = \; (x+y)(x+y)(x+y)$$

Each term has form $x^k y^{3-k}$

▷ total degree of a term in $(x+y)^3$ is 3

$$(x+y)^3 \; = \; xxx + xxy + xyx + xyy + yxx + yxy + yyx + yyy$$

x^3 : number of ways to choose 3 factors for x is : $1 = \binom{3}{3}$

x^2y : number of ways to choose 2 factors for x is : $3 = \binom{3}{2}$

xy^2 : number of ways to choose 1 factors for x is : $3 = \binom{3}{1}$

y^3 : number of ways to choose 0 factors for x is : $1 = \binom{3}{0}$

The Binomial Theorem

$$(x+y)^n = \underbrace{(x+y)(x+y)\dots(x+y)}_{n \text{ factors}}$$

Each term has form $x^k y^{n-k}$

▷ total degree of a term in $(x+y)^n$ is n

For $0 \leq k \leq n$ coefficient of $x^k y^{n-k}$ is $\binom{n}{k}$

The Binomial Theorem

The Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

This is why $\binom{n}{k}$ is called the binomial coefficient

The Binomial Theorem

The Binomial Theorem

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

We can prove many theorems using the binomial theorem

The Binomial Theorem

The number of subsets of an n-element set is 2^n

By sum rule, the number of subsets $= \sum_{k=0}^{n} \binom{n}{k}$

The Binomial Theorem

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Substitute $x = 1$ in the binomial formula

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k$$

The Binomial Theorem

The number of even-sized subsets of a set is equal to the number of odd-sized subsets

Equivalently, $$\sum_{\substack{k=0 \\ k \text{ odd}}}^{n} \binom{n}{k} = \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k}$$

Substitute $x = -1$ in the binomial formula

$$0 = (1-1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k = \sum_{\substack{k=0 \\ k \text{ odd}}}^{n} -\binom{n}{k} + \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \binom{n}{k}$$