Counting

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Imdad ullah Khan

Permutation

A permutation or arrangement of n objects is an ordering of the objects

The number of permutations of n objects is $n!$

An r-permutation is an ordering or arrangement of r out of n objects

The number of r-permutations of n objects is

$$n(n-1)(n-2)\cdots(n-(r-1))$$

Permutations with repetition

How many r-permutations with repetitions from n objects are there?

We discussed it earlier in detail

- How many 5-digits postal codes are there?
- How many different license plates can be made?
- How many strings of length r can be formed from the uppercase letters of the English alphabet?

Think of filing r boxes experiment and apply the product rule

The number of r-permutations from n objects with repetition is n^r

Combinations

An r-combination is a (unordered) subset of size r of n objects

The number of r-combinations of n objects is

$$\frac{P(n,r)}{r!} = \frac{n!}{(n-r)!r!} = \binom{n}{r}$$

$\binom{n}{r}$: n choose r (binomial coefficient)

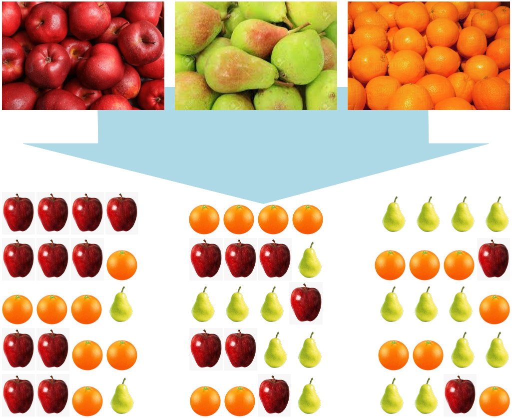
(the number of r-subsets of an n-element set)

Combinations with repetition

How many ways are there to select 4 pieces of fruits from a bowl of apples(A), oranges(O), and pears(P)?

Given:

- order of pieces selection does not matter
- only the type and not the individual pieces matters
- there are at least four pieces of each type of fruit (unlimited)



Combinations with repetition

How many ways are there to select 4 pieces of fruits from a bowl of apples(A), oranges(O), and pears(P)? Given:

- order of pieces selection does not matter
- only the type and not the individual pieces matters ▷ identical
- there are at least four pieces of each type of fruit (unlimited)

Number of 4-combinations with repetition from 3-element (multi)set = 15

- $\{A, A, A, A\}$
- $\{A, A, A, O\}$
- $\{O, O, O, P\}$
- $\{A, A, O, O\}$
- $\{A, A, O, P\}$
- $\{O, O, O, O\}$
- $\{A, A, A, P\}$
- $\{P, P, P, A\}$
- $\{A, A, P, P\}$
- $\{O, O, A, P\}$
- $\{P, P, P, P\}$
- $\{O, O, O, A\}$
- $\{P, P, P, O\}$
- $\{O, O, P, P\}$
- $\{P, P, A, O\}$

Combinations with repetition

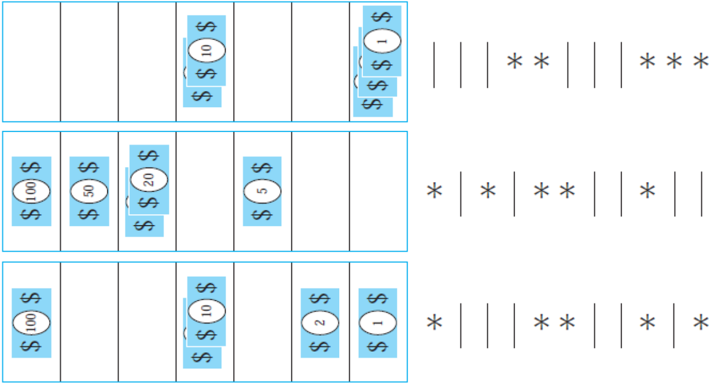
Cash Box with Seven Types of Bills.

How many ways are there to select 5 bills from a cashbox of $1, $2, $5, $10, $20, $50, and $100 bills? Given:

- order of chosen bills does not matter
- only the type and not the individual bill matters
- there are at least five bills of each type

Combinations with repetition

Represent the selection of 5 bills as an arrangement of 5 $*$'s and 6 $|$'s



Examples of Ways to Select Five Bills.

Combinations with repetition

Represent the selection of 5 bills as an arrangement of 5 $*$'s and 6 $\Big|$'s

- $\Big|$ denotes a separator of the cashbox separating the types of bills. 6 separators to represent 7 types of bills
- $*$ denotes a bill. $*$ between two separators represents the selected number of bills of the type

select 5 bills $\implies$ arrange 6 $\Big|$ and 5 $*$ in a row with a total of 11 positions

$\therefore$ number of ways to select the 5 bills $=$ number of ways to select the positions of the 5 stars from the 11 positions

$\implies$ Solution: num of unordered selections of 5 objects from a set of 11

$$= \binom{11}{5} = \frac{11!}{5!6!} = 462$$

Combinations with repetition

There are $\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$ r-combinations from a set with n elements when repetitions of elements are allowed

Suppose that a cookie shop has 4 different kinds of cookies. In how many different ways can 6 cookies be chosen? *(similar assumptions as above)*

Solution:

number of 6-combinations of a set with 4 elements.

From above rule this equals:

$$\binom{4+6-1}{6} = \binom{9}{6} = 84$$

Combinations with repetition

ICP 11-30

How many non-negative integral solutions are there for the equation

$$x + y + z = 100$$

$$\overbrace{* \ * \ * \ * \ \cdots \ * \ * \ *}^{x \text{ stars}} \ \Big| \ \overbrace{* \ * \ * \ * \ * \ \cdots \ * \ * \ *}^{y \text{ stars}} \ \Big| \ \overbrace{* \ * \ * \ * \ * \ * \ \cdots \ * \ * \ * \ *}^{z=100-x-y \text{ stars}}$$

Choose 100 $*$ from a multiset of $*, *, *$