Counting

- Introduction and Applications
- Sum and Product Rule
- The Complement Rule
- Inclusion-Exclusion Principle
- The Pigeon-Hole Principle
- Permutations and Combinations
- Combinatorial Proofs
- Permutation and Combinations with Repetitions

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Binomial Identities: Algebraic Proof

ICP 11-27 Prove that

$$\binom{n}{k} = \binom{n}{n-k}$$

Algebraic Proof:

$$\binom{n}{n-k} = \frac{n!}{(n-(n-k))!(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}$$

Binomial Identities: Combinatorial Proof

Show that both sides of the identity count the same objects

$$\binom{n}{k} = \binom{n}{n-k}$$

Combinatorial Proof:

Counting Task: Select k-subset of an n-set

- LHS counts the number of ways to select k objects out of n
- RHS counts the number of ways to reject $n-k$ objects out of n

Both count the number of ways to partition n objects into k selected and $n-k$ rejected objects

Binomial Identities: Algebraic Proof

ICP 11-28 Prove that

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

Algebraic Proof:

$$k\binom{n}{k} = \frac{k \cdot n!}{(n-k)!k!} = \frac{n!}{(n-k)!(k-1)!}$$

$$= \frac{n \cdot (n-1)!}{((n-1)-(k-1))!(k-1)!} = n\binom{n-1}{k-1}$$

Binomial Identities: Combinatorial Proof

Show that both sides of the identity count the same objects

$$k\binom{n}{k} = n\binom{n-1}{k-1} \implies \binom{k}{1}\binom{n}{k} = \binom{n}{1}\binom{n-1}{k-1}$$

Combinatorial Proof:

Counting Task: Select k players out of n players and a captain

- LHS: Select k players and among them select a captain
- RHS: Select a captain and from remaining select $k-1$ players

Both count number of ways to select team with captain

Binomial Identities: Algebraic Proof

ICP 11-29 Prove that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Algebraic Proof:

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \frac{(n-1)!}{(n-1-k)!k!} + \frac{(n-1)!}{(n-k)!(k-1)!}$$

$$= \frac{(n-k)(n-1)! \;+\; (k)(n-1)!}{(n-k)!\,(k)!}$$

$$= \frac{(n)!}{(n-k)!(k)!} = \binom{n}{k}$$

Binomial Identities: Combinatorial Proof

Show that both sides of the identity count the same objects

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Combinatorial Proof:

Counting Task: Select k-subset of an n-set

- LHS counts the number of k-subsets of an n-set
- Fix an element x in the n-set. RHS counts two types of k-subsets.

  Those $\underbrace{\text{containing the element } x}_{\binom{n-1}{k-1}}$ and $\underbrace{\text{those not containing } x}_{\binom{n-1}{k}}$

Pascal Identity

This last identity is called the Pascal Identity, we will discuss it in detail

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$