# Counting

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# Permutation

How many ways to arrange the letters $A$, $B, C$, and $D$ ?

$$4 \times 3 \times 2 \times 1$$

How many 5 digits codes are there with no repetition ?

$$10 \times 9 \times 8 \times 7 \times 6$$

How many ways to arrange $n$ letters ?

$$n \times (n-1) \times (n-2) \times \cdots\cdots\cdots \times 3 \times 2 \times 1$$

How many $r$ digits codes are there with no repetition from $n$ characters ?

$$n \times (n-1) \times (n-2) \times \cdots \times (n-(r-1))$$

# Permutation

A permutation or arrangement of $n$ objects is an ordering of the objects

The number of permutations of $n$ objects is $n!$

An $r$-permutation is an ordering or arrangement of $r$ out of $n$ objects

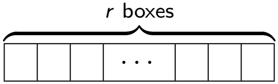
The number of $r$-permutations of $n$ objects is

$$n(n-1)(n-2)\cdots(n-(r-1))$$

# Permutation

$P(n, r)$ : Number of $r$-permutation of $n$ objects

Number of ways to permute, order or arrange $r$ out of $n$ objects




$$P(n, r) = n(n-1)(n-2)\cdots(n-(r-1))$$

$$= \frac{n!}{(n-r)!}$$

# Ordered Versus Unordered

I have 5 students, $\{A, B, C, D, E\}$ and I give the prizes \$500 and \$300 to two students.

How many choices do I have?

Order of selection matters here, $A, B$ is different than $B, A$

$$P(5, 2)$$

# Ordered Versus Unordered

I have 5 students, $\{A, B, C, D, E\}$ and I give the grade $A$ to two students.

How many choices do I have?

Order of selection does not matter, $A, B$ is the same as $B, A$

| | |
|---|---|
| $\{A, B\}$ | $AB, BA$ |
| $\{A, C\}$ | $AC, CA$ |
| $\{A, D\}$ | $AD, DA$ |
| $\{A, E\}$ | $AE, EA$ |
| $\{B, C\}$ | $BC, CB$ |
| $\{B, D\}$ | $BD, DB$ |
| $\{B, E\}$ | $BE, EB$ |
| $\{C, D\}$ | $CD, DC$ |
| $\{C, E\}$ | $CE, EC$ |
| $\{D, E\}$ | $DE, ED$ |

Number of 2 letter arrangements from $\{A, B, C, D, E\}$

$$P(5,2) = \frac{5!}{3!} = 20$$

But these are ordered pairs

Each unordered pair is listed twice

Each unordered pair is listed 2! times

Number of unordered pairs $= 20/2! = 10$

# Ordered Versus Unordered

**ICP 11-24** How many subsets of size 5 are there for the set $\{0,1,2,3,4,5,6,7,8,9\}$?

Number of ordered 5 length sequences that can be formed from $\{0,1,2,3,4,5,6,7,8,9\}$

$$P(10,5) = \frac{10!}{(10-5)!} = 30240$$

Number of arrangements/ordering of 5 digits is

$$P(5,5) = 5! = 120$$

Each subset is counted 120 times. Number of (unordered) subsets

$$\frac{P(10,5)}{5!} = \frac{10!}{(10-5)!5!} = \frac{30240}{120} = 252$$

# Combinations

An $r$-combination is a (unordered) subset of size $r$ of $n$ objects

The number of $r$-combinations of $n$ objects is

$$\frac{P(n,r)}{r!} = \frac{n!}{(n-r)!r!} = \binom{n}{r}$$

$\binom{n}{r}$ : $n$ choose $r$ (binomial coefficient)

(the number of $r$-subsets of an $n$-element set)

# Combinations

Number of $r$-combinations of $n$ objects is $\binom{n}{r} = \dfrac{n!}{(n-r)!r!}$

Number of 0-subsets of $\{a, b, c, d\} = \binom{4}{0} = \dfrac{4!}{(4-0)!0!} = 1$

Number of 1-subsets of $\{a, b, c, d\} = \binom{4}{1} = \dfrac{4!}{(4-1)!1!} = 4$

Number of 2-subsets of $\{a, b, c, d\} = \binom{4}{2} = \dfrac{4!}{(4-2)!2!} = 6$

Number of 3-subsets of $\{a, b, c, d\} = \binom{4}{3} = \dfrac{4!}{(4-3)!3!} = 4$

Number of 4-subsets of $\{a, b, c, d\} = \binom{4}{4} = \dfrac{4!}{(4-4)!4!} = 1$

# Combinations

**ICP 11-25** How many 3-bit sequences with *two* 0's and *one* 1

- 3 possible positions to put the first 0
- 2 remaining positions to put the second 0
- 1 remaining to put the 1

$$3 \times 2 \times 1$$

$$001, 010, 100$$

Choose the 2 positions for 0's and the 1 is forced

$$\binom{3}{2} = 3$$

# Combinations

**ICP 11-26** How many 7-bit sequences with 3 0's and 4 1's

Choose 3 positions for 0's and the 1's are forced

$$\binom{7}{3}$$

Equivalently,

Choose 4 positions for 1's and the 0's are forced

$$\binom{7}{4}$$

# Counting with Combinations

Poker: 52 Card Deck, 5 card hands

4 possible suits: ♥ ♠ ♦ ♣

13 possible values: $2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A$

- Pair: set of two cards of the same rank
- Straight: 5 cards of consecutive rank
- Flush: set of 5 cards with the same suit
- Straight Flush: a straight and a flush
- 4 of a kind: 4 cards of the same value
- Full House: 3 of one value and 2 of another
- Flush: a flush, but not a straight
- Straight: a straight, but not a flush
- 3 of a kind: 3 of the same value, but not a full house or 4 of a kind
- 2 Pairs: 2 pairs, but not 4 of a kind or full house

# Counting with Combinations

4 of a kind: 4 cards of the same value

How many different Poker hands with 4 of a kind are there?

- 13 choices for the value of the four cards
- 12 choices for the value of the fifth card
- 4 choices for the suit of the fifth card

$$13 \times 12 \times 4 = 624$$

Equivalently,

- 13 choices for the value of the four cards
- 48 choices for the fifth card

$$13 \times 48 = 624$$

# Counting with Combinations

Full House: 3 of one value and 2 of another

How many different Poker hands with Full House are there?

- 13 choices for the value of the three cards
- $\binom{4}{3}$ choices for the suits of the three cards
- 12 choices for the value of the pair of cards
- $\binom{4}{2}$ choices for the suits of the pair

$$13 \times \binom{4}{3} \times 12 \times \binom{4}{2} = 3,744$$

# Counting with Combinations

A school committee of 8 faculty members has to be formed, subject to

- The committee should have 5 and 3 members from the EE and CS dept
- EE department has 30 and CS has 20 members

How many different committees can be made?

$$\binom{30}{5} \times \binom{20}{3}$$