Counting

- Introduction and Applications
- Sum and Product Rule
- The Complement Rule
- Inclusion-Exclusion Principle
- The Pigeon-Hole Principle
- Permutations and Combinations
- Combinatorial Proofs
- Permutation and Combinations with Repetitions

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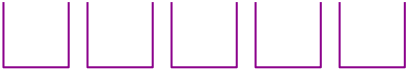
Pigeonhole Principle

If there are more pigeons

than there are pigeon holes,



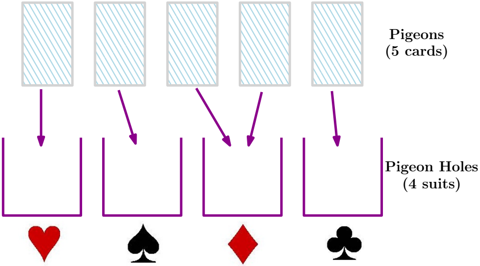
then some pigeon hole must have more than one pigeons

This is obvious! Less obvious! Where would this ever be useful?

- Probability of collision by hash functions
- Proof that there cannot be a lossless compression of some size

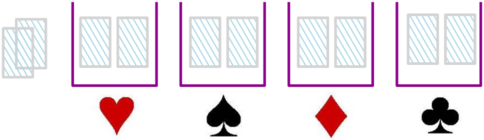
Pigeonhole Principle

Any set of 5 cards must have at least 2 cards of the same suit



Pigeonhole Principle

Any set of 10 cards must have at least how many of the same suit



Number of cards of the same suit $\geq \left\lceil \frac{10}{4} \right\rceil$

Generalized Pigeonhole Principle

ICP 11-16 Prove that in any set of 13 people at least two have the same birth month

- Pigeons: The 13 peoples' birth months
- Pigeon holes: The 12 months in a year

Generalized Pigeonhole Principle

6 exam time slots per day in the exam week (7 days)

- No two simultaneous exams in a room
- 300 courses exams have to be done in one week

How many rooms are needed at the minimum?

$$\left\lceil \frac{300}{42} \right\rceil = 8$$

General Pigeonhole Principle

If $\mathbf{n}$ pigeons are placed into $\mathbf{h}$ holes,

then some hole must have at least $\left\lceil \frac{\mathbf{n}}{\mathbf{h}} \right\rceil$ pigeons

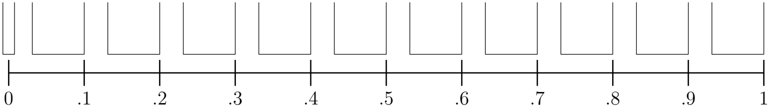
Pigeonhole Principle

In any set of 12 numbers in the interval $[0, 1]$, there are at least 2 numbers x, y such that $|x - y| \leq 0.1$

Pigeons: The 12 numbers

Pigeonholes: The intervals

$$\{0\}, (0, 0.1], (0.1, 0.2], (0.2, 0.3], \ldots, (0.9, 1]$$



Pigeonhole Principle

- Party of n people
- Some handshaking takes place.
- A pair either shake hands once or do not.

At least 2 people shook the same number of hands

Pigeons: The $n \geq 2$ people

Pigeon Holes: Possible number of handshakes

$$\{0, 1, 2, \ldots, n-1\}$$

Pigeonhole Principle

At least two people shook the same number of hands

Pigeons: The $n \geq 2$ people

Pigeon Holes: Possible number of handshakes

$$\{0, 1, 2, \ldots, n-1\}$$

$$|\text{pigeon holes}| = |\text{pigeons}|$$

Case 1: Someone shook $n-1$ hands

No one has 0 handshakes. n pigeons, $n-1$ pigeon holes

Case 2: No one shook $n-1$ hands

No one has $n-1$ handshakes. n pigeons, $n-1$ pigeon holes

Generalized Pigeonhole Principle

ICP 11-17

Five departments, each with 100 students.

How many students should be selected so that from some department at least 6 students are selected?

$$\left\lceil \frac{n}{5} \right\rceil = 6 \quad \text{or} \quad \frac{n}{5} > 5 \quad \text{or} \quad n > 25$$

Generalized Pigeonhole Principle

1000 people go from LHR to ISB in 22 buses each with 60 seats

ICP 11-18 Some bus must carry more than at least x people

- Pigeons: The 1000 filled seats
- Pigeon holes: The 22 buses

ICP 11-19 Some bus must have at least y empty seats

- Pigeons: The empty seats
- Pigeon holes: The 22 buses

Generalized Pigeonhole Principle

ICP 11-20 Prove that any set of eleven 2-digit numbers must have two elements whose digits have the same absolute difference

$\triangleright$ e.g. 37 and 48, $|3-7| = |4-8|$

- Pigeons: The 11 absolute differences of digits of each number
- Pigeon holes: The 10 possible absolute differences of digits

Generalized Pigeonhole Principle

ICP 11-21 Choose three different digits from 1 to 9. Write all ordering of those digits. Prove that among all the 3-digit numbers written that way there are two whose difference is a multiple of 500.

- Pigeons: Number of integers that can be made this way $(9 \times 8 \times 7)$

  ▷ Actually their remainders when divided by 500

- Pigeon holes: Number of possible remainders when divided by 500

Generalized Pigeonhole Principle

ICP 11-22 Prove that among any $n+1$ numbers selected from the set $S = \{1, 2, 3, \ldots, 2n\}$ there are two that have no factor in common

▷ Hint: Use the lemma *"if x divides y, then it does not divide $y+1$"*

- Pigeons: The $n+1$ chosen numbers
- Pigeon holes: The n pairs of consecutive numbers in S