Counting

- Introduction and Applications
- Sum and Product Rule
- The Complement Rule
- Inclusion-Exclusion Principle
- The Pigeon-Hole Principle
- Permutations and Combinations
- Combinatorial Proofs
- Permutation and Combinations with Repetitions

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Inclusion-Exclusion Principle

I have 32 students in one section and 35 students in the other section. Suppose I give the grade A to one student.

How many choices do I have in total?

Number of choices $= 32 + 35$

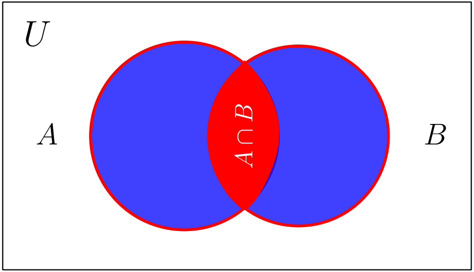
I have 32 students in Disc. Math. and 35 students in Prog. course. Suppose I give the grade A to one student.

How many choices do I have in total?

Number of choices $= 32 + 35$

Students in both courses are counted twice

Inclusion-Exclusion Principle



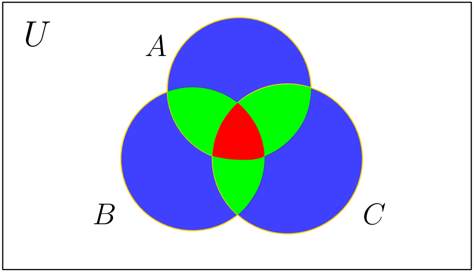
Inclusion-Exclusion Principle

$$\left|A \cup B\right| = \left|A\right| + \left|B\right| - \left|A \cap B\right|$$

ICP 11-11 Let $A = \{2, 4, 7, 9\}$ and $B = \{1, 3, 7, 8, 9, 5\}$.

What is $\left|A \cup B\right|$?

Inclusion-Exclusion Principle



Inclusion-Exclusion Principle

$$
\begin{aligned}
\left|A \cup B \cup C\right| = \;& \left|A\right| + \left|B\right| + \left|C\right| \\
& - \left|A \cap B\right| - \left|A \cap C\right| - \left|B \cap C\right| \\
& + \left|A \cap B \cap C\right|
\end{aligned}
$$

Inclusion-Exclusion Principle

Inclusion-Exclusion Principle

$$\begin{aligned} |A \cup B \cup C| = \; & |A| + |B| + |C| \\ & - |A \cap B| - |A \cap C| - |B \cap C| \\ & + |A \cap B \cap C| \end{aligned}$$

ICP 11-12 $A = \{2, 4, 7, 9\}$, $B = \{3, 7, 8, 9, 5\}$, and $C = \{1, 4, 5, 6, 9\}$

What is $|A \cup B \cup C|$?

General Inclusion-Exclusion Principle

General Inclusion-Exclusion Principle

$$\begin{aligned}
\left|A_1 \cup A_2 \cup \ldots \cup A_n\right| &= \sum_{i} \left|A_i\right| \\
&- \sum_{i \neq j} \left|A_i \cap A_j\right| \\
&+ \sum_{i \neq j \neq k} \left|A_i \cap A_j \cap A_k\right| \\
&\ldots \\
&+ (-1)^{n-1} \left|A_1 \cap A_2 \cap \ldots \cap A_n\right|
\end{aligned}$$

General Inclusion-Exclusion Principle

How many integers between 1 and 300 (inclusive) are divisible by 5 or 7?

- $M_5 = \{a \in \mathbb{Z} \ : \ 1 \leq a \leq 300 \ \wedge \ 5|a \ \}$
- $M_7 = \{a \in \mathbb{Z} \ : \ 1 \leq a \leq 300 \ \wedge \ 7|a \ \}$

$$|M_5| = \left\lfloor \frac{300}{5} \right\rfloor \qquad |M_7| = \left\lfloor \frac{300}{7} \right\rfloor \qquad |M_5 \cap M_7| = \left\lfloor \frac{300}{35} \right\rfloor$$

$$|M_5 \cup M_7| = |M_5| + |M_7| - |M_5 \cap M_7|$$

General Inclusion-Exclusion Principle

ICP 11-13

How many integers between 1 and 300 (inclusive) are divisible by $3, 5$ or 7?

- $M_3 = \{a \in \mathbb{Z} \,:\, 1 \leq a \leq 300 \,\wedge\, 3|a\}$
- $M_5 = \{a \in \mathbb{Z} \,:\, 1 \leq a \leq 300 \,\wedge\, 5|a\}$
- $M_7 = \{a \in \mathbb{Z} \,:\, 1 \leq a \leq 300 \,\wedge\, 7|a\}$

1. $|M_3| = ?\quad |M_5| = ?\quad |M_7| = ?$
2. $|M_3 \cap M_5| = ?\quad |M_5 \cap M_7| = ?\quad |M_3 \cap M_7| = ?$
3. $|M_3 \cap M_5 \cap M_7| = ?$

$$|M_3 \cup M_5 \cup M_7| = ?$$

General Inclusion-Exclusion Principle

ICP 11-14

How many integers between 1 and 300 (inclusive) are divisible by $3, 5$ but not 7?

- $M_3 = \{a \in \mathbb{Z} \ : \ 1 \leq a \leq 300 \ \wedge \ 3|a \}$
- $M_5 = \{a \in \mathbb{Z} \ : \ 1 \leq a \leq 300 \ \wedge \ 5|a \}$
- $M_7 = \{a \in \mathbb{Z} \ : \ 1 \leq a \leq 300 \ \wedge \ 7|a \}$

1. $|M_3| = ?$ $\quad |M_5| = ?$ $\quad |\overline{M_7}| = ?$
2. $|M_3 \cap M_5| = ?$ $\quad |M_5 \cap \overline{M_7}| = ?$ $\quad |M_3 \cap \overline{M_7}| = ?$
3. $|M_3 \cap M_5 \cap \overline{M_7}| = ?$

$$|M_3 \cup M_5 \cup \overline{M_7}| = ?$$

General Inclusion-Exclusion Principle

ICP 11-15

How many integers between 1 and 300 (inclusive) are divisible by 7 but by neither 3 nor 5?

General Inclusion-Exclusion Principle

Find the number of passwords made from the characters

$$\{a, b, f, g, h, l, m, o, t, u\}$$

- Length of password is 10 but no repetition
- No password contain the word 'gulf', 'math' or 'boat'.

Count invalid passwords!

Passwords containing 'gulf' — Treat 'gulf' as a block! 7!

'gulf' and 'math' — 'gulf' and 'math' as blocks! 4!

'boat' and 'math' — 'boat' and 'math' as blocks! 0

$$10! - 7! - 7! - 7! + 4! + 4!$$

Analyzing a school report

90 students in DM: 60 are boys. $\triangleright\ |D| = 90, |D \cap B| = 60$

60 students in CP; 32 are boys. $\triangleright\ |P| = 60, |B \cap P| = 32$

56 students are regist. in Eco; 36 are boys and 34 are also regist. in CP. $\triangleright\ |E| = 56, |B \cap E| = 36, |E \cap P| = 34$

30 boys are registered in CP and Eco. $\triangleright\ |B \cap E \cap P| = 30$

D : students in DM (all students). B : boys.
$P \subseteq D$: students in CP $E \subseteq D$: students in Eco.

Find G : those girls who are only taking DM

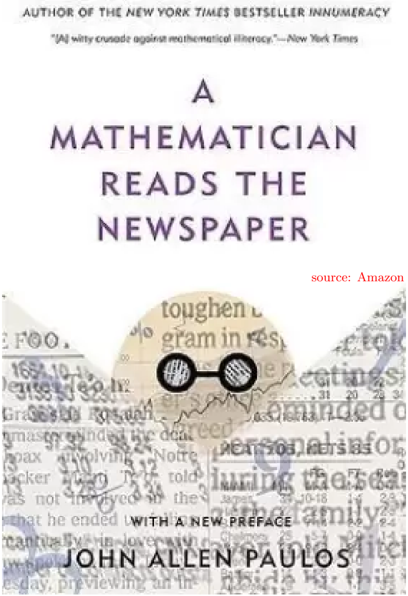
$G = D \setminus (B \cup E \cup P)$

$|G| = |D \setminus (B \cup E \cup P)| = |D| - |B \cup E \cup P| =$

$|D| - |B| - |E| - |P| + |B \cap E| + |B \cap P| + |E \cap P| - |B \cap E \cap P|$

$= 90 - 60 - 56 - 60 + 36 + 32 + 34 - 30 = -14$

Analyzing a school report



source: Amazon

General Inclusion-Exclusion Principle

Some common and useful bounds on cardinalities that follow from the Inclusion-Exclusion Principle

Theorem

Suppose A and B are subsets of a finite universal set U. Then

1. $|A \cup B| \; = \; |A| + |B| - |A \cap B|$
2. $|A \cap B| \; \leq \; \min\left\{|A|, |B|\right\}$
3. $|A \cup B| \; \geq \; \max\left\{|A|, |B|\right\}$
4. $|A \setminus B| \; = \; |A| - |A \cap B| \; \geq \; |A| - |B|$
5. $|\overline{A}| \; = \; |U| - |A|$
6. $|A \oplus B| \; = \; |A \cup B| - |A \cap B| \; = \; |A| + |B| - 2|A \cap B| = |A \setminus B| + |B \setminus A|$