Counting

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- Permutations and Combinations
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Sum Rule

Sum Rule

If a task can be done either in one of n_1 ways or in one of n_2 different ways, then there are $n_1 + n_2$ ways to do the task

Product Rule

Suppose you have 9 shirts and 6 pairs of pants

How many choices do you have for an outfit?

$$9 \times 6 = 54$$

Product Rule

Suppose you have 9 shirts, 6 pairs of pants and 4 ties

How many choices do you have for an outfit?

$$9 \times 6 \times 4 = 216$$

Product Rule

Product Rule

Suppose a procedure can be broken down into two successive tasks.

- If there are n_1 ways to do the first task and
- for each way of doing the first task, there are n_2 ways to do the second task,

then there are $n_1 \times n_2$ ways to do the procedure

Product Rule

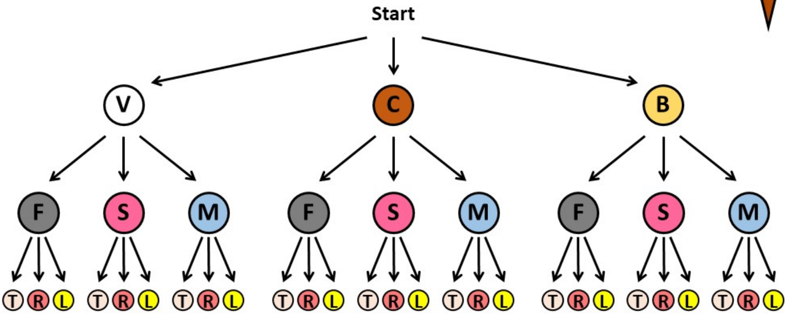
Flavours
Vanilla
Chocolate
Banana

Toppings
Flake
Sprinkles
Marshmallows

Sauce
Toffee
Raspberry
Lemon

We can **map** the possibilities.

What would happen to the total amount of possibilities if we can choose a sauce as well?



To find the total possibilities we can **multiply** each independent option.

3 × 3 × 3 = 27 total possibilities

source: https://www.goteachmaths.co.uk/

Product Rule

Set theoretic version of the Product Rule

$$\left|A \times B\right| = \left|A\right| \times \left|B\right|$$

In general,

$$\left|A_1 \times A_2 \times \ldots \times A_n\right| = \left|A_1\right| \times \left|A_2\right| \times \cdots \times \left|A_n\right|$$

Product Rule

How many ways to arrange the letters A and B ?

Break the procedure of arranging A and B into two successive tasks

1. choose first letter ▷ 2 ways to do it
2. choose second letter ▷ For each way for task-1, 1 way to do it

Number of ways to arrange A and B is (2×1)

$AB \quad BA$

Product Rule

ICP 11-3 How many ways to arrange the letters A, B, and C ?

Break the procedure of arranging A, B, and C into 3 successive tasks

1. choose first letter ▷ 3 ways to do it
2. choose second letter ▷ For each way for task-1, 2 ways to do it
3. choose third letter ▷ For each way for task-1 & 2, 1 way to do it

$$ABC, \quad ACB, \quad BAC, \quad BCA, \quad CAB, \quad CBA$$

Product Rule

ICP 11-4 How many ways to arrange the letters A, B, C, and D ?

$ABCD$	$BACD$	$CABD$	$DABC$
$ABDC$	$BADC$	$CADB$	$DACB$
$ACBD$	$BCAD$	$CBAD$	$DBAC$
$ACDB$	$BCDA$	$CBDA$	$DBCA$
$ADBC$	$BDAC$	$CDAB$	$DCAB$
$ADCB$	$BDCA$	$CDBA$	$DCBA$

ICP 11-5 How many ways to arrange n letters ?

Number of Letters	Ways to arrange them
1	$1 = 1!$
2	$2 = 2!$
3	$6 = 3!$
4	$24 = 4!$

Number of ways to arrange n letters is $n!$

Product Rule

4 CS students $\{a, b, c, d\}$ and 3 EE students $\{x, y, z\}$ want to make teams of 2, with 1 CS and 1 EE student in each team

How many different teams can be made?

Break team-making into 2 successive tasks

1. choose CS student ▷ 4 ways to do it
2. choose EE student ▷ For each way for task-1, 3 ways to do it

Number of different teams: $4 \times 3 = 12$

(a,x)	(a,y)	(a,z)
(b,x)	(b,y)	(b,z)
(c,x)	(c,y)	(c,z)
(d,x)	(d,y)	(d,z)

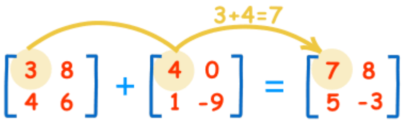
Product Rule

ICP 11-6 Suppose you have to choose a project from 4 S/W projects AND a project from 5 research projects.

How many choices do you have?

$$4 \times 5 = 20$$

Product Rule



Matrix Addition

Algorithm Addition of two $m \times n$ matrices A and B, return $C = A + B$

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for $i = 1$ to m **do**

  for $j = 1$ to n **do**

    $C[i][j] \leftarrow A[i][j] + B[i][j]$

return C

How many addition operations are performed?

$$m \times n$$

Product Rule

A license plate contains 3 letters and 4 digits.

How many different license plates are possible?

L	*E*	*B*	5	7	0	0

1st Place – 26 choices
2nd Place – 26 choices
3rd Place – 26 choices
4th Place – 10 choices
5th Place – 10 choices
6th Place – 10 choices
7th Place – 10 choices

$$(26)^3 \times (10)^4$$

Product Rule

ICP 11-7 How many 5 digits Postal Codes are there?

0	8	8	5	4

1st Place – 10 choices ▷ (0,1,2,3,4,5,6,7,8,9)
2nd Place – 10 choices ▷ (0,1,2,3,4,5,6,7,8,9)
3rd Place – 10 choices ▷ (0,1,2,3,4,5,6,7,8,9)
4th Place – 10 choices ▷ (0,1,2,3,4,5,6,7,8,9)
5th Place – 10 choices ▷ (0,1,2,3,4,5,6,7,8,9)

$$(10)^5$$

Product Rule

ICP 11-8 How many 5 digits Postal Codes are there with no repetition?

0	8	9	3	7

1st Place – 10 choices ▷ (0,1,2,3,4,5,6,7,8,9)

2nd Place – 9 choices ▷ (1,2,3,4,5,6,7,8,9)

3rd Place – 8 choices ▷ (1,2,3,4,5,6,7,9)

4th Place – 7 choices ▷ (1,2,3,4,5,6,7)

5th Place – 6 choices ▷ (1,2,4,5,6,7)

$$(10 \times 9 \times 8 \times 7 \times 6)$$

Product Rule

ICP 11-9 How many passwords can be made with the following rules?

- Can contain digits and case sensitive English letters
- Must be 5 to 7 characters long
- Must begin with a letter

P_5 := set of length 5 passwords

P_6 and P_7 are similarly defined

$$|P_5| = 52 \times 62^4$$
$$|P_6| = 52 \times 62^5$$
$$|P_7| = 52 \times 62^6$$

P_5, P_6, P_7 make partition of all valid passwords

A 'valid' password in exactly one of P_5, P_6, or P_7

By the sum rule, total number of valid passwords $= |P_5| + |P_6| + |P_7|$