Induction

- Principle of Mathematical Induction
- Proofs by Induction
- Strong Induction
- Well Ordering Principle

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Proof by Induction

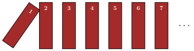
A proposition about non-negative integers, $\forall\ n\ P(n)$ is a sequence of propositions (dominoes)

$$P(0), P(1), P(2), \ldots, P(n), P(n+1), \ldots$$

Establish two facts

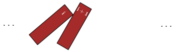
- Prove $P(0)$

  the first domino falls

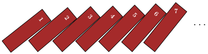


- Prove $\forall k \geq 0, P(k) \rightarrow P(k+1)$

  if a domino falls, then the next domino also falls



Conclude that $P(n)$ is true for all n



Principle of Mathematical Induction

$$\Big[\, P(0) \wedge \forall k \geq 0 \left[P(k) \rightarrow P(k+1) \right] \Big] \longrightarrow \forall n \geq 0\, P(n)$$

Strong Induction

Fundamental Theorem of Arithmetic

Every integer ≥ 2 can be written as product of prime(s)

$P(n) :=$ n can be written as product of primes

Fundamental Theorem of Arithmetic

$\forall\ n \geq 2\ \ P(n)$ $\triangleright$ n can be factored into prime(s)

Strong Induction

Fundamental Theorem of Arithmetic

$\forall\ n \geq 2\ \ P(n)$ $\qquad \triangleright$ n can be factored into prime(s)

Base Case: 2 is prime, so $P(2)$ is true

Inductive Hypothesis: Assume $P(n)$ is true

Inductive Step: Using $P(n)$ prove $P(n+1)$

Strong Induction

Fundamental Theorem of Arithmetic

$\forall\ n \geq 2\ \ P(n)$ $\quad\triangleright$ n can be factored into prime(s)

IH: Assume $P(22)$ is true. $22 = 2 \cdot 11$

Inductive Step: Using $P(22)$ prove $P(23)$

23 is prime, so we don't need $P(22)$ $\quad\triangleright * \rightarrow T = T$

Inductive Step: Using $P(23)$ prove $P(24)$

$24 = 6 \cdot 4$, how can we use $P(23)$?

Strong Induction

Principle of Mathematical Induction

$$\Big[\, P(0) \wedge \forall k \geq 0 \,\big[P(k) \rightarrow P(k+1)\big]\,\Big] \longrightarrow \forall n \geq 0\, P(n)$$

Proof using Induction

- *Basis Step:* Prove $P(0)$ is true
- *IH:* Assume $P(n)$
- *Inductive Step:* Using $P(n)$, prove $P(n+1)$

Principle of Strong Mathematical Induction

$$\Big[\, P(0) \wedge \forall k \geq 0 \,\big[\forall\, 0 \leq i \leq k\; P(i) \rightarrow P(k+1)\big]\,\Big] \longrightarrow \forall n \geq 0\, P(n)$$

Proof using Strong Induction

- *Basis Step:* Prove $P(0)$ is true
- *IH:* Assume $P(k)$ is true for all $1 \leq k \leq n$
- *IS:* Using $\forall\; k \leq n\; P(k)$, prove $P(n+1)$

Strong Induction

Fundamental Theorem of Arithmetic

$\forall\ n \geq 2\ \ P(n)$ $\triangleright$ n can be factored into prime(s)

IH: Assume $P(22)$ is true. $22 = 2 \cdot 11$

Inductive Step: Using $P(22)$ prove $P(23)$

23 is prime, so we don't need $P(22)$ $\triangleright$ $* \to T = T$

Inductive Step: Using $P(23)$ prove $P(24)$

$24 = 6 \cdot 4$, how can we use $P(23)$?

Strong Induction

Fundamental Theorem of Arithmetic

$\forall\ n \geq 2\ \ P(n)$ $\qquad\triangleright$ n can be factored into prime(s)

The proof is by strong induction on n

Base Case: 2 is prime, so $P(2)$ is true

IH: Assume $P(k)$ is true for $1 \leq k \leq n$

Inductive Step: Using **IH** prove $P(n+1)$

- If $(n+1)$ is prime, $P(n+1)$ is true
- else $(n+1)\ =\ a \cdot b$
  - $a, b \leq n$
  - by **IH** both a and b factor into primes
  - so $(n+1)$ factors into primes (grand product of factors of a and b)

Strong Induction

Fundamental Theorem of Arithmetic

$\forall\ n \geq 2\ \ P(n)$ $\qquad \triangleright$ n can be factored into prime(s)

Later we will prove that

Fundamental Theorem of Arithmetic

Every integer ≥ 2 can be uniquely written as product of prime(s)

Primes are the building blocks of natural numbers

ICP 10-10

Show that every positive integer n can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers in

$$\{1, 2, 4, 8, 16, 32, \dots\}$$

[Hint: For the inductive step, consider the cases when $k+1$ is even and odd. When it is even, then $(k+1)/2$ is an integer.]