Induction

- Principle of Mathematical Induction
- Proofs by Induction
- Strong Induction
- Well Ordering Principle

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Proof by Induction

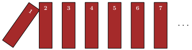
A proposition about non-negative integers, $\forall\ n\ P(n)$ is a sequence of propositions (dominoes)

$$P(0), P(1), P(2), \ldots, P(n), P(n+1), \ldots$$

Establish two facts

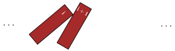
- Prove $P(0)$

  the first domino falls

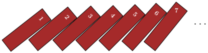


- Prove $\forall k \geq 0, P(k) \rightarrow P(k+1)$

  if a domino falls, then the next domino also falls



Conclude that $P(n)$ is true for all n



Principle of Mathematical Induction

$$\Big[\, P(0) \wedge \forall k \geq 0 \left[P(k) \rightarrow P(k+1)\right] \Big] \longrightarrow \forall n \geq 0\, P(n)$$

Proof by Induction

Theorem

Number of subsets of an n element set is 2^n

Basis Step: $n = 0$:

Number of subsets of an empty set is $1 = 2^0$

Proof by Induction

Theorem

Number of subsets of an n element set is 2^n

IH: Assume it is true for *'any $n-1$ elements set'*

For an n-element set S, fix an element $x \in S$ and Let $S' = S \setminus \{x\}$

$\mathcal{P}(S) =$ subsets not containing x $\bigcup$ subsets containing x

$\triangleright$ Subsets not containing x are subsets of S'

By **IH**, number of subsets of $S' = 2^{n-1}$ $\triangleright \because |S'| = n-1$

$\triangleright$ Subset containing x is $\{x\} \cup$ a subset of S'

- number of subsets not containing $x = 2^{n-1}$
- number of subsets containing $x = 2^{n-1}$

$|\mathcal{P}(S)| = 2(2^{n-1}) = 2^n$

Proof by Induction

Theorem

An ATM with only \$3 and \$5 bills can generate any amount $n \geq 8$

Basis Step: $n = 8$ dollar. A \$3 bill and a \$5 bill

Induction Step: Need to prove that given that the ATM can generate n dollar it can generate $n + 1$ dollar amount

Case 1: Output of \$$n$ contains a \$5 bill

Remove a \$5 bill and add two \$3 bills

Case 2: Output of \$$n$ contains no \$5 bill, it must have $\geq$ three \$3 bills

Remove three of \$3 and add two \$5 bills

Proof by Induction

ICP 10-7 Prove that

Theorem

An ATM with only \$3 and \$7 bills can generate any amount $n \geq 12$

Basis Step: $n = 12$ dollar. $\triangleright$ $4 \times \$3$

Induction Step: Prove if ATM can generate $\$n$, then it can generate $\$n + 1$

Case 1: Output of $\$n$ contains $\geq$ two \$3 bills

Case 2: Output of $\$n$ contains $\leq$ one \$3 bill, it must have $\geq$ two \$7 bills

Loop Invariant

FINDMAX()

$max \leftarrow 0$
$for(i = 0; i < n; i++)$
    $if(A[i] > max)$
        $max \leftarrow A[i]$

A is an array of n positive integers

Claim

$max =$ the maximum of A

Loop Invariant

A proposition that remains true throughout the loop

$$P: \text{ After iteration } i \quad max = \text{ maximum of } A[0 \ldots i]$$

- True after iteration $i = 0$
- If true after iteration i, then true after iteration $i + 1$

Proof by Induction: Importance of Basis Step

Theorem

$n(n+1)$ is an odd integer

Proof: Assume it is true for n, i.e.

IH: $n(n+1)$ is an odd number

Given **IH**, show $(n+1)(n+2)$ is an odd number

$$(n+1)(n+2) = n(n+1) + 2(n+1)$$

By the IH, $n(n+1)$ is odd, $2(n+1)$ is even so

$$(n+1)(n+2) \text{ is an odd number}$$

$2(3) = 6$ and $3(4) = 12$

Basis Step is missing

Proof by Induction: Importance of Basis Step

Theorem

For every positive integer n, $\sum_{i=1}^{n} i \; = \; (n + \frac{1}{2})^2/2$

"Basis Step:" The formula is true for $n = 1$.

"Inductive Step:" Suppose $\displaystyle\sum_{i=1}^{n} i = \left(n + \frac{1}{2}\right)^2 /2$. Then

$\displaystyle\sum_{i=1}^{n+1} i = \sum_{i=1}^{n} i + (n+1)$. By the inductive hypothesis

$$\sum_{i=1}^{n+1} i \; = \; \left(n + \tfrac{1}{2}\right)^2/2 + n + 1 \; = \; \left(n^2 + n + \tfrac{1}{4}\right)/2 + n + 1 \; =$$
$$\left(n^2 + 3n + \tfrac{9}{4}\right)/2 \; = \; \left(n + \tfrac{3}{2}\right)^2/2 \; = \; \left((n+1) + \tfrac{1}{2}\right)^2/2.$$

ICP 10-8 What is wrong with this proof?

Proof by Induction: Importance of Inductive Step

Theorem

All horses are of the same color

$P(n) :=$ Any set of n horses have the same color

Theorem

$\forall\ n\ P(n)$ *is true*

Base Case: A single horse is the same color as itself, so $P(1)$ is true

IH: $P(n)$ is true, any n horses have the same color

Inductive Step: Using $P(n)$ prove $P(n+1)$

Proof by Induction: Importance of Inductive Step

Theorem

$\forall\ n\ P(n)$ *(Any n horses have the same color)*

IH: Any n horses have the same color

Inductive Step: Using $P(n)$ prove $P(n+1)$

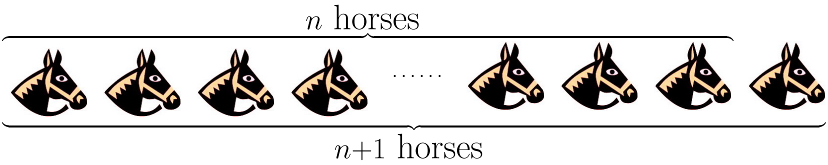
Proof by Induction: Importance of Inductive Step

Theorem

$\forall\ n\ P(n)$ *(Any n horses have the same color)*

IH: Any n horses have the same color

Inductive Step: Using $P(n)$ prove $P(n+1)$



- By **IH**, the first n horses are of the same color
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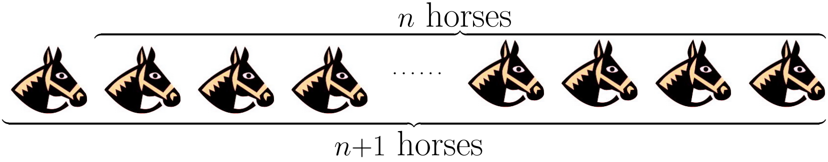
Proof by Induction: Importance of Inductive Step

Theorem

$\forall\ n\ P(n)$ *(Any n horses have the same color)*

IH: Any n horses have the same color

Inductive Step: Using $P(n)$ prove $P(n+1)$



- By **IH**, the first n horses are of the same color
- By **IH**, the last n horses are of the same color
- The common horses among the first last n horses can have one color
- Thus all $n+1$ have the same color

Proof by Induction: Importance of Inductive Step

Theorem

$\forall\ n\ P(n)$ *(Any n horses have the same color)*

ICP 10-9 What is wrong with the above proof?