Induction

- Principle of Mathematical Induction
- Proofs by Induction
- Strong Induction
- Well Ordering Principle

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Proof by Induction

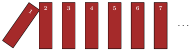
A proposition about non-negative integers, $\forall\ n\ P(n)$ is a sequence of propositions (dominoes)

$$P(0), P(1), P(2), \ldots, P(n), P(n+1), \ldots$$

Establish two facts

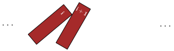
- Prove $P(0)$

  the first domino falls

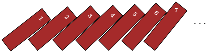


- Prove $\forall k \geq 0, P(k) \rightarrow P(k+1)$

  if a domino falls, then the next domino also falls



Conclude that $P(n)$ is true for all n



Principle of Mathematical Induction

$$\big[\, P(0) \wedge \forall k \geq 0 \left[P(k) \rightarrow P(k+1) \right] \big] \longrightarrow \forall n \geq 0\, P(n)$$

Proof by Induction

Theorem

$$P(n)\ :\ \sum_{i=0}^{n} i \ =\ \frac{n(n+1)}{2}$$

ICP 10-1

Formally rewrite to make a (explicit) statement about all positive integers

Theorem

$$\forall n \in \mathbb{N}\ \ \sum_{i=0}^{n} i \ =\ \frac{n(n+1)}{2} \qquad or \qquad \forall n \in \mathbb{N}\ \ P(n)$$

Proof by Induction

Theorem

$$\forall n \in \mathbb{N} \ \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

Proof Outline:

1. The proof is by induction on k
2. Basis Step: Prove that $P(k)$ is true for $k = 0$
3. Induction Step: Prove that for all k, $P(k) \rightarrow P(k+1)$
4. Conclude that $\forall\, n \in \mathbb{N}\ P(n)$ is true

Proof by Induction

Theorem

$$\forall n \in \mathbb{N} \ \ \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

Proof:

Basis Step: need to prove that $P(0)$ is true

$$P(0) : \quad \sum_{i=0}^{0} i = \frac{0(0+1)}{2}$$

This is clearly true, because both sides are equal to 0

Proof by Induction

Theorem

$$\forall n \in \mathbb{N} \ \sum_{i=0}^{n} i = \frac{n(n+1)}{2}$$

Proof:

Inductive Step: need to prove that $\forall k \ P(k) \rightarrow P(k+1)$ is true

Inductive Hypothesis (IH): Assume that $P(k)$ is true i.e. $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$

Given IH, show that $P(k+1)$ is true, i.e, $\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}$

Proof by Induction

Inductive Step: need to prove that $\forall k \;\; P(k) \rightarrow P(k+1)$ is true

Inductive Hypothesis (IH): Assume that $P(k)$ is true i.e. $\sum_{i=0}^{k} i = \frac{k(k+1)}{2}$

Given IH, show that $P(k+1)$ is true, i.e, $\sum_{i=0}^{k+1} i = \frac{(k+1)(k+2)}{2}$

$$\sum_{i=0}^{k+1} i = \sum_{i=0}^{k} i + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1) \qquad \triangleright \text{ by the Inductive Hypothesis}$$

$$= (k+1)(\tfrac{k}{2}+1) = \frac{(k+1)(k+2)}{2} \implies P(k+1) \text{ is true} \qquad \square$$

Proof by Induction

Theorem

The sum of first n positive odd integers is n^2

ICP 10-2

Restate it formally to make it a statement about all positive integers

Theorem

$$\forall n \in \mathbb{Z}^+ \quad \sum_{i=1}^{n}(2i-1) \;=\; n^2$$

Proof by Induction

Theorem

$$\forall n \in \mathbb{Z}^+ \ \ \sum_{i=1}^{n}(2i-1) \ = \ n^2$$

ICP 10-3 What will be the basis step?

Proof: The proof is by induction on n.

Basis Step: $n = 1$

$$P(1) \ : \ \sum_{i=1}^{1}(2i-1) = 1 = 1^2$$

Clearly true!

Proof by Induction

Theorem

$$\forall n \in \mathbb{Z}^+ \quad \sum_{i=1}^{n}(2i-1) \;=\; n^2$$

Inductive Hypothesis (IH): Assume $P(n)$ is true, i.e.

$$\sum_{i=1}^{n}(2i-1) \;=\; n^2$$

Inductive Step: Given IH, show that $P(n+1)$ is true,

$$\sum_{i=1}^{n+1}(2i-1) \;=\; (n+1)^2$$

Proof by Induction

Inductive Step: Given IH, show that $P(n+1)$ is true,

$$\sum_{i=1}^{n+1}(2i-1) = (n+1)^2$$

$$\begin{aligned}
\sum_{i=1}^{n+1}(2i-1) &= \sum_{i=1}^{n}(2i-1) + 2(n+1) - 1 \\
&= n^2 + 2(n+1) - 1 && \triangleright \text{ by the inductive hypothesis} \\
&= n^2 + (2n+1) \\
&= (n+1)^2 && \square
\end{aligned}$$

Proof by Induction

Theorem

Sum of first n powers of 2 is $2^n - 1$

Start with $2^0 = 1$

ICP 10-4 Formally, restate as a statement about all positive integers?

Theorem

$$\forall n \in \mathbb{N} \quad \sum_{i=0}^{n-1} 2^i \; = \; 2^n - 1$$

Proof by Induction

Theorem

$$\forall n \in \mathbb{N} \quad \sum_{i=0}^{n-1} 2^i = 2^n - 1$$

Proof: Basis Step: Rule: Sum of nothing is 0.

$$\sum_{i=0}^{-1} 2^i = 0 \quad \text{and} \quad 2^0 - 1 = 0$$

Inductive Step:

$$\sum_{i=0}^{n} 2^i = \sum_{i=0}^{n-1} 2^i + 2^n = 2^n - 1 + 2^n = 2^{n+1} - 1$$

Proof by Induction

ICP 10-5 Prove that

Theorem

The sum of squares of first n positive integers is $\dfrac{n(n+1)(2n+1)}{6}$

Proof: Proof by induction on n

Basis Step: $n = 0$ $n = 1$ $1^2 = \dfrac{1(1+1)(2(1)+1)}{6}$

Inductive Step: Prove

$$\underbrace{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}_{\text{inductive hypothesis}} \implies \sum_{i=1}^{n+1} i^2 = \frac{(n+1)(n+2)(2n+3)}{6}$$

Proof by Induction

ICP 10-6 Prove that

Theorem

The sum of cubes of first n positive integers is $\left(\frac{n(n+1)}{2}\right)^2$

Proof: Proof by induction on n

Basis Step: $n = 1$ $\quad 1^3 = \left(\frac{1(1+1)}{2}\right)^2$

Inductive Step: Prove

$$\underbrace{\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2}_{\text{inductive hypothesis}} \implies \sum_{i=1}^{n+1} i^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2$$