Cardinality

- List Representation of Functions and Cardinality of Finite Sets
    - Properties of Functions as Lists
- Cardinality of Infinite Sets
- Countable and Uncountable Infinite Sets
    - Diagonalization

Imdad ullah Khan

Cardinality of finite sets

Georg Cantor (1874) defined the equivalence of cardinalities of infinite sets

For finite sets X and Y, $|X| = |Y|$ iff there is a bijection $f : X \to Y$

For ~~finite~~ sets X and Y, $|X| = |Y|$ iff there is a bijection $f : X \mapsto Y$

$|X| = |Y|$, if X and Y can be placed in a **one-to-one correspondence**

Countability

A set S is countable if it is either finite or has the same cardinality as $\mathbb{N}$

S is countable if it can be placed in a **one-to-one correspondence** with $\mathbb{N}$

S is countable in the following sense

If we count (write, print, list) one element of S per 'second', *then any particular element of S will be counted after a finite time*

This means we can list element of S like

$$a_1, a_2, a_3, a_4, a_5, \cdots$$

Note: We do not say that the whole set will be printed

Countability

A set S is **countable** if it is either finite or has the same cardinality as $\mathbb{N}$

Are the following sets countable (countably infinite)?

ICP 9-15 $\mathbb{Z}$

ICP 9-16 $\mathbb{O}$ and $\mathbb{E}$, odd and even integers

ICP 9-17 Integer powers of 2

ICP 9-18 Integer powers of other integers

ICP 9-19 Squares, cubes and any power of integers

Countability

A set S is **countable** if it is either finite or has the same cardinality as $\mathbb{N}$

We showed that powers of 2, powers of integers, $\mathbb{Z}$ etc. are countable

How about $\mathbb{Q}^+$, the set of $+$ve rational numbers

Rational numbers are very *dense*

Theorem

Between any two rational numbers there is another rational number

Countability

Arrange the positive rational numbers (p/q) in a table, such that row q contains all rationals with denominator q

$$\begin{array}{ccccccc}
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \frac{6}{1} & \cdots \\
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \frac{6}{2} & \cdots \\
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \frac{6}{3} & \cdots \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} & \frac{6}{4} & \cdots \\
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \frac{5}{5} & \frac{6}{5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Countability

Counting them row-wise or column wise does not yield countability

For instance the number $4/3$ is never counted (printed)

$$
\begin{array}{ccccccccccccc}
\frac{1}{1} & \longrightarrow & \frac{2}{1} & \longrightarrow & \frac{3}{1} & \longrightarrow & \frac{4}{1} & \longrightarrow & \frac{5}{1} & \longrightarrow & \frac{6}{1} & & \cdots \\
\frac{1}{2} & & \frac{2}{2} & & \frac{3}{2} & & \frac{4}{2} & & \frac{5}{2} & & \frac{6}{2} & & \cdots \\
\frac{1}{3} & & \frac{2}{3} & & \frac{3}{3} & & \frac{4}{3} & & \frac{5}{3} & & \frac{6}{3} & & \cdots \\
\frac{1}{4} & & \frac{2}{4} & & \frac{3}{4} & & \frac{4}{4} & & \frac{5}{4} & & \frac{6}{4} & & \cdots \\
\frac{1}{5} & & \frac{2}{5} & & \frac{3}{5} & & \frac{4}{5} & & \frac{5}{5} & & \frac{6}{5} & & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Countability

Counting them row-wise or column wise does not yield countability

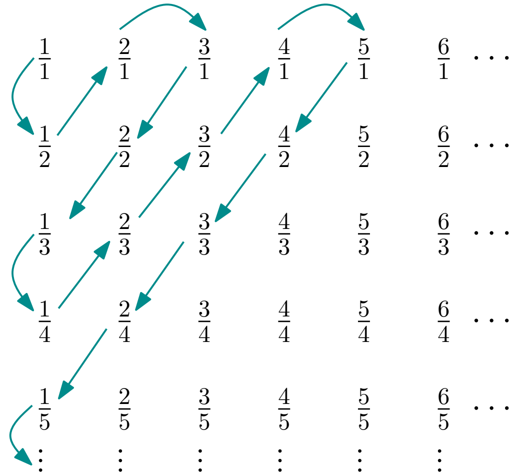
For instance the number $4/3$ is never counted (printed)

$$
\begin{array}{ccccccc}
\frac{1}{1} & \frac{2}{1} & \frac{3}{1} & \frac{4}{1} & \frac{5}{1} & \frac{6}{1} & \cdots \\
\downarrow & & & & & & \\
\frac{1}{2} & \frac{2}{2} & \frac{3}{2} & \frac{4}{2} & \frac{5}{2} & \frac{6}{2} & \cdots \\
\downarrow & & & & & & \\
\frac{1}{3} & \frac{2}{3} & \frac{3}{3} & \frac{4}{3} & \frac{5}{3} & \frac{6}{3} & \cdots \\
\downarrow & & & & & & \\
\frac{1}{4} & \frac{2}{4} & \frac{3}{4} & \frac{4}{4} & \frac{5}{4} & \frac{6}{4} & \cdots \\
\downarrow & & & & & & \\
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} & \frac{5}{5} & \frac{6}{5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}
$$

Countability

Counting them in the dove-tailing fashion shows countability

Any number $^{p}/_{q}$ is counted after *'finite'?* steps



Countability

Are all infinite sets of the same size (countable)?

No

Cantor invented a very important technique,

DIAGNOLIZATION

to show how to find bigger infinity

Uncountable set

The set $\mathbb{I}$ of real numbers between 0 and 1 is not countable

We prove it by contradiction and draw contradiction using *'diagnolization'*

Assume that $\mathbb{I} = [0, 1]$ is countable

Then by definition there exists a bijection $f : \mathbb{N} \mapsto \mathbb{I}$
and we can list the elements of $\mathbb{I}$

List element of $\mathbb{I}$ in a table form as follows:

Diagnolization

The set $\mathbb{I}$ of real numbers between 0 and 1 is not countable

Assume that $\mathbb{I} = [0,1]$ is countable

Then there exists a bijection $f : \mathbb{N} \mapsto \mathbb{I}$ $\triangleright$ and we can list/count $\mathbb{I}$

List elements of $\mathbb{I}$ in a table form as follows:

$$
\begin{aligned}
f(1) &= r_1 = 0.\, d_{11}\, d_{12}\, d_{13}\, d_{14}\, d_{15} \ldots \\
f(2) &= r_2 = 0.\, d_{21}\, d_{22}\, d_{23}\, d_{24}\, d_{25} \ldots \\
f(3) &= r_3 = 0.\, d_{31}\, d_{32}\, d_{33}\, d_{34}\, d_{35} \ldots \\
f(4) &= r_4 = 0.\, d_{41}\, d_{42}\, d_{43}\, d_{44}\, d_{45} \ldots \\
f(5) &= r_5 = 0.\, d_{51}\, d_{52}\, d_{53}\, d_{54}\, d_{55} \ldots \\
&\vdots
\end{aligned}
$$

Diagnolization

L	1	2	3	4	5	...
r_1	d_{11}	d_{12}	d_{13}	d_{14}	d_{15}	...
r_2	d_{21}	d_{22}	d_{23}	d_{24}	d_{25}	...
r_3	d_{31}	d_{32}	d_{33}	d_{34}	d_{35}	...
r_4	d_{41}	d_{42}	d_{43}	d_{44}	d_{45}	...
r_5	d_{51}	d_{52}	d_{53}	d_{54}	d_{55}	...

Diagnolization

L	1	2	3	4	5	$\cdots$
r_1	d_{11}	d_{12}	d_{13}	d_{14}	d_{15}	$\ldots$
r_2	d_{21}	d_{22}	d_{23}	d_{24}	d_{25}	$\ldots$
r_3	d_{31}	d_{32}	d_{33}	d_{34}	d_{35}	$\ldots$
r_4	d_{41}	d_{42}	d_{43}	d_{44}	d_{45}	$\ldots$
r_5	d_{51}	d_{52}	d_{53}	d_{54}	d_{55}	$\ldots$

Diagnolization

L	1	2	3	4	5	⋯
r_1	d_{11}					⋯
r_2		d_{22}				⋯
r_3			d_{33}			⋯
r_4				d_{44}		⋯
r_5					d_{55}	⋯

Construct a new number $S_L \in \mathbb{I} = [0,1]$, $S_L = 0.\ C_1,\ C_2,\ C_3,\ C_4,\ C_5,\ \ldots$

$$C_i \ = \begin{cases} 8 & if\ \ d_{ii} = 7 \\ 7 & if\ \ d_{ii} \neq 7 \end{cases}$$

Diagnolization

L	1	2	3	4	5	$\cdots$
r_1	$d_{11} \neq C_1$	C_2	C_3	C_4	C_5	$\ldots$
r_2		d_{22}				$\ldots$
r_3			d_{33}			$\ldots$
r_4				d_{44}		$\ldots$
r_5					d_{55}	$\ldots$

Construct a new number $S_L \in \mathbb{I} = [0,1]$, $S_L = 0.\ C_1,\ C_2,\ C_3,\ C_4,\ C_5,\ \ldots$

$$C_i = \begin{cases} 8 & if\ \ d_{ii} = 7 \\ 7 & if\ \ d_{ii} \neq 7 \end{cases}$$

$S_L \neq r_1$, $\because$ by design $C_1 \neq d_{11}$ $\quad\triangleright$ different than r_1 in at least 1st digit

Diagnolization

L	1	2	3	4	5	$\cdots$
r_1	d_{11}					$\dots$
r_2	C_1	$d_{22} \neq C_2$	C_3	C_4	C_5	$\dots$
r_3			d_{33}			$\dots$
r_4				d_{44}		$\dots$
r_5					d_{55}	$\dots$

Construct a new number $S_L \in \mathbb{I} = [0,1]$, $S_L = 0.\ C_1,\ C_2,\ C_3,\ C_4,\ C_5,\ \dots$

$$C_i = \begin{cases} 8 & if\ \ d_{ii} = 7 \\ 7 & if\ \ d_{ii} \neq 7 \end{cases}$$

$S_L \neq r_2$, $\because$ by design $C_2 \neq d_{22}$ ▷ different than r_2 in at least 2nd digit

Diagnolization

L	1	2	3	4	5	$\cdots$
r_1	d_{11}					$\ldots$
r_2		d_{22}				$\ldots$
r_3	C_1	C_2	$d_{33} \neq C_3$	C_4	C_5	$\ldots$
r_4				d_{44}		$\ldots$
r_5					d_{55}	$\ldots$

Construct a new number $S_L \in \mathbb{I} = [0,1]$, $S_L = 0.\ C_1,\ C_2,\ C_3,\ C_4,\ C_5,\ \ldots$

$$C_i = \begin{cases} 8 & if\ d_{ii} = 7 \\ 7 & if\ d_{ii} \neq 7 \end{cases}$$

$S_L \neq r_3$, $\because$ by design $C_3 \neq d_{33}$ ▷ different than r_3 in at least 3rd digit

Diagnolization

L	1	2	3	4	5	$\cdots$
r_1	d_{11}					$\ldots$
r_2		d_{22}				$\ldots$
r_3			d_{33}			$\ldots$
r_4	C_1	C_2	C_3	$d_{44} \neq C_4$	C_5	$\ldots$
r_5					d_{55}	$\ldots$

Construct a new number $S_L \in \mathbb{I} = [0,1]$, $S_L = 0.\ C_1,\ C_2,\ C_3,\ C_4,\ C_5,\ \ldots$

$$C_i = \begin{cases} 8 & \textit{if}\ d_{ii} = 7 \\ 7 & \textit{if}\ d_{ii} \neq 7 \end{cases}$$

$S_L \neq r_4$, $\because$ by design $C_4 \neq d_{44}$ $\triangleright$ different than r_4 in at least 4th digit

Diagnolization

L	1	2	3	4	5	$\cdots$
r_1	d_{11}					$\ldots$
r_2		d_{22}				$\ldots$
r_3			d_{33}			$\ldots$
r_4				d_{44}		$\ldots$
r_5	C_1	C_2	C_3	C_4	$d_{55} \neq C_5$	$\ldots$

Construct a new number $S_L \in \mathbb{I} = [0,1]$, $S_L = 0.\ C_1,\ C_2,\ C_3,\ C_4,\ C_5,\ \ldots$

$$C_i = \begin{cases} 8 & if\ d_{ii} = 7 \\ 7 & if\ d_{ii} \neq 7 \end{cases}$$

$S_L \neq r_5$, $\because$ by design $C_5 \neq d_{55}$ ▷ different than r_5 in at least 5th digit

Diagnolization

L	1	2	3	4	5	$\cdots$
r_1	d_{11}					$\ldots$
r_2		d_{22}				$\ldots$
r_3			d_{33}			$\ldots$
r_4				d_{44}		$\ldots$
r_5	C_1	C_2	C_3	C_4	$d_{55} \neq C_5$	$\ldots$

Construct a new number $S_L \in \mathbb{I} = [0, 1]$, $S_L = 0.\ C_1,\ C_2,\ C_3,\ C_4,\ C_5,\ \ldots$

$$C_i = \begin{cases} 8 & if\ \ d_{ii} = 7 \\ 7 & if\ \ d_{ii} \neq 7 \end{cases}$$

$S_L \neq r_i$, $\because$ by design $C_i \neq d_{ii}$ $\triangleright$ different than r_i in at least ith digit

The list L is incomplete and the assumed function is not a bijection

Diagnolization

The set $\mathbb{I}$ of real numbers between 0 and 1 is not countable

Assume that $\mathbb{I} = [0,1]$ is countable

Then there exists a bijection

$f : \mathbb{N} \mapsto \mathbb{I}$

and we can list/count $\mathbb{I}$

$$
\begin{aligned}
r_1 &= 0 \cdot 4653365973230847 \cdots \\
r_2 &= 0 \cdot 2283691758160085 \cdots \\
r_3 &= 0 \cdot 2470709451841855 \cdots \\
r_4 &= 0 \cdot 7974760149589142 \cdots \\
r_5 &= 0 \cdot 7042107951435447 \cdots \\
r_6 &= 0 \cdot 3004775241525619 \cdots \\
r_7 &= 0 \cdot 5337390062326107 \cdots \\
r_8 &= 0 \cdot 1831822059411461 \cdots \\
r_9 &= 0 \cdot 0690882743970627 \cdots \\
r_{10} &= 0 \cdot 7540952087698266 \cdots \\
r_{11} &= 0 \cdot 6986696148745598 \cdots \\
r_{12} &= 0 \cdot 6486231101480499 \cdots \\
r_{13} &= 0 \cdot 1927265701330231 \cdots \\
r_{14} &= 0 \cdot 6302255117726757 \cdots \\
r_{15} &= 0 \cdot 1970378555725766 \cdots \\
r_{16} &= 0 \cdot 3871428845438795 \cdots \\
S_L &= 0 \cdot 7787787778877877 \cdots
\end{aligned}
$$

Uncountable Set

The set $\mathbb{I}$ of real numbers between 0 and 1 is not countable

Thus all infinite sets are not of the same cardinality

There are more than one type of infinities