Cardinality

- List Representation of Functions and Cardinality of Finite Sets
  - Properties of Functions as Lists
- Cardinality of Infinite Sets
- Countable and Uncountable Infinite Sets
  - Diagonalization

Imdad ullah Khan

Cardinality of finite sets

- Cardinality of a finite set X is the number of distinct elements in X
- X and Y have the same cardinality, if $|X| = |Y|$
- If $f : X \mapsto Y$ is a bijection and X and Y are finite sets, then $|X| = |Y|$

For finite sets X and Y, $|X| = |Y|$ iff there is a bijection $f : X \mapsto Y$

Proof of the if part

"if $|X| = |Y|$, then there is a bijection"

- Fix any ordering on X and Y
- $f(x_i) = y_i$ is a bijection

Cardinality of infinite sets

$\mathbb{Z}^+ =$ set of positive integers

$\mathbb{Z}^- =$ set of negative integers

$\mathbb{Z}^+ = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots\}$

$\mathbb{Z}^- = \{-1, -2, -3, -4, -5, -6, -7, -8, -9, -10, \ldots\}$

Which one is bigger?

Cardinality of infinite sets

$\mathbb{O}$ = set of positive odd integers

$\mathbb{E}$ = set of positive even integers

$\mathbb{O} = \{1, 3, 5, 7, 9, 11, 13, 15, 17, \ldots\}$

$\mathbb{E} = \{2, 4, 6, 8, 10, 12, 14, 16, 18, \ldots\}$

Which one is bigger?

Cardinality of infinite sets

$\mathbb{E}$ = set of positive even integers

$\mathbb{N}$ = set of natural numbers

$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 \ldots\}$

$\mathbb{E} = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, \ldots\}$

$\mathbb{E} \subset \mathbb{N}$

Which one is bigger?

Cardinality of infinite sets

$\mathbb{N}$ = set of natural numbers

$\mathbb{S}$ = set of perfect squares

$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13 \ldots\}$

$\mathbb{S} = \{1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169 \ldots\}$

$\mathbb{S} \subset \mathbb{N}$

Which one is bigger?

Cardinality of infinite sets

$\mathbb{N}$ = set of natural numbers

$\mathbb{C}$ = set of perfect cubes

$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, \ldots\}$

$\mathbb{C} = \{1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331 \ldots\}$

$\mathbb{C} \subset \mathbb{N}$

Which one is bigger?

Cardinality of infinite sets

$\mathbb{N}$ = set of natural numbers

$\mathbb{X}$ = set of powers of 2

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \ldots\}$$

$$\mathbb{X} = \{2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, \ldots\}$$

$$\mathbb{X} \subset \mathbb{N}$$

Which one is bigger?

Cardinality of finite sets

For finite sets X and Y, $|X| = |Y|$ iff there is a bijection $f : X \mapsto Y$

Georg Cantor (1874) defined the equivalence of cardinalities of infinite sets

For ~~finite~~ sets X and Y, $|X| = |Y|$ iff there is a bijection $f : X \mapsto Y$

$|X| = |Y|$, if X and Y can be placed in a **one-to-one correspondence**

Cardinality of infinite sets

$\mathbb{Z}^+$ = positive integers $\mathbb{Z}^-$ = negative integers

$\mathbb{Z}^+ = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots\}$

$\mathbb{Z}^- = \{-1, -2, -3, -4, -5, -6, -7, -8, -9, -10, \ldots\}$

$\mathbb{Z}^+$	$\mathbb{Z}^-$	$\mathbb{Z}^-$
1	-1	$-(1)$
2	-2	$-(2)$
3	-3	$-(3)$
4	-4	$-(4)$
5	-5	$-(5)$
6	-6	$-(6)$
7	-7	$-(7)$
8	-8	$-(8)$
9	-9	$-(9)$

ICP 9-9 Design a bijection $f : \mathbb{Z}^+ \mapsto \mathbb{Z}^-$

$f(x) = -x$

$|\mathbb{Z}^+| = |\mathbb{Z}^-|$

Cardinality of infinite sets

$\mathbb{O}$ = positive odd integers $\qquad$ $\mathbb{E}$ = positive even integers

$\mathbb{O} = \{1, 3, 5, 7, 9, 11, 13, 15, 17, \ldots\}$

$\mathbb{E} = \{2, 4, 6, 8, 10, 12, 14, 16, 18, \ldots\}$

$\mathbb{O}$	$\mathbb{E}$	$\mathbb{E}$
1	2	$1+1$
3	4	$3+1$
5	6	$5+1$
7	8	$7+1$
9	10	$9+1$
11	12	$11+1$
13	14	$13+1$
15	16	$15+1$
17	18	$17+1$

ICP 9-10 Design a bijection $f : \mathbb{O} \mapsto \mathbb{E}$

$f(x) = x + 1$

$|\mathbb{O}| = |\mathbb{E}|$

Cardinality of infinite sets

$\mathbb{N}$ = natural numbers. $\mathbb{E}$ = set of +ve even integers

$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 \ldots\}$

$\mathbb{E} = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, \ldots\}$

$\mathbb{N}$	$\mathbb{E}$	$\mathbb{E}$
1	2	2(1)
2	4	2(2)
3	6	2(3)
4	8	2(4)
5	10	2(5)
6	12	2(6)
7	14	2(7)
8	16	2(8)
9	18	2(9)

$\mathbb{E} \subset \mathbb{N}$

ICP 9-11 Design a bijection $f : \mathbb{N} \mapsto \mathbb{E}$

$f(x) = 2(x)$

$|\mathbb{N}| = |\mathbb{E}|$

Cardinality of infinite sets

$\mathbb{N}$ = set of natural numbers $\qquad$ $\mathbb{S}$ = set of perfect squares

$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 \ldots\}$

$\mathbb{S} = \{1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, \ldots\}$

$\mathbb{N}$	$\mathbb{S}$	$\mathbb{S}$
1	1	1^2
2	4	2^2
3	9	3^2
4	16	4^2
5	25	5^2
6	36	6^2
7	49	7^2
8	64	8^2
9	81	9^2

$\mathbb{S} \subset \mathbb{N}$

ICP 9-12 Design a bijection $f : \mathbb{N} \mapsto \mathbb{S}$

$f(x) = x^2$

$|\mathbb{N}| = |\mathbb{S}|$

Cardinality of infinite sets

$\mathbb{N}$ = set of natural numbers $\qquad$ $\mathbb{C}$ = set of perfect cubes

$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots\}$

$\mathbb{C} = \{1, 8, 27, 64, 125, 216, 343, 512, 729, 1000 \ldots\}$

$\mathbb{N}$	$\mathbb{C}$	$\mathbb{C}$
1	1	1^3
2	8	2^3
3	27	3^3
4	64	4^3
5	125	5^3
6	216	6^3
7	343	7^3
8	512	8^3
9	729	9^3

$\mathbb{C} \subset \mathbb{N}$

ICP 9-13 $f : \mathbb{N} \mapsto \mathbb{C}$

$f(x) = x^3$

$|\mathbb{N}| = |\mathbb{C}|$

Cardinality of infinite sets

$\mathbb{N}$ = set of natural numbers $\qquad$ $\mathbb{X}$ = set of powers of 2

$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \ldots\}$

$\mathbb{X} = \{2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, 2048, \ldots\}$

$\mathbb{N}$	$\mathbb{X}$	$\mathbb{X}$
1	2	2^1
2	4	2^2
3	8	2^3
4	16	2^4
5	32	2^5
6	64	2^6
7	128	2^7
8	256	2^8
9	512	2^9
10	1024	2^{10}

$\mathbb{X} \subset \mathbb{N}$

ICP 9-14 $\quad f : \mathbb{N} \mapsto \mathbb{X}$

$f(x) = 2^x$

$|\mathbb{N}| = |\mathbb{X}|$

Cardinality of infinite sets

We showed that

- |integer powers of 2 and other integers| $= |\mathbb{N}|$
- |powers of all integers| $= |\mathbb{N}|$
- $|\mathbb{Z}| = |\mathbb{N}|$

"*size*/2 = *size*". Surprised!

> *I see it, but I don't believe it!*
>
> George Cantor (in a letter to Dedekind, 1877)

This notion of cardinality enables us to reason about infinity

Cardinality of infinite sets

It does make some sense though!

$\infty + 1 = \infty$

$\infty \pm \text{finite number } = \infty$

$2 \cdot \infty = \infty$

$\infty \cdot \text{finite number } = \infty$

$\infty / 2 = \infty$

$$\frac{\infty}{\text{finite number}} = \infty$$