Cardinality

- List Representation of Functions and Cardinality of Finite Sets
  - Properties of Functions as Lists
- Cardinality of Infinite Sets
- Countable and Uncountable Infinite Sets
  - Diagonalization

Imdad ullah Khan

Function

Let X and Y be two sets. A function f maps **each** element of X to **exactly one** element of Y

Let $f : X \mapsto Y$ and let $f(x) = y$

- X is the domain of f
- Y is the codomain of f
- y is the image of x
- x is the pre-image of y
- Range of f: set of images of every $x \in X$

Function: List Representation

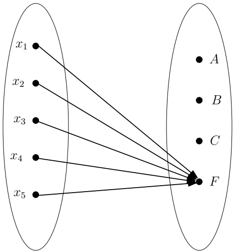
Let X and Y be two sets. A function f maps **each** element of X to **exactly one** element of Y

Let X be the domain with its **elements ordered** $x_1, x_2 \ldots,$

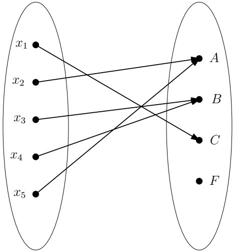
$f : X \mapsto Y$ can be represented as a list $\quad f(x_1), f(x_2), f(x_3), \ldots$

- Images of $x_1, x_2, \ldots$ listed in the order of X

Function: List Representation



x_i	$f(\cdot)$
x_1	F
x_2	F
x_3	F
x_4	F
x_5	F



x_i	$f(\cdot)$
x_1	C
x_2	A
x_3	B
x_4	B
x_5	A

Properties of functions as lists

A function $f : X \mapsto Y$ is **one-to-one** (or **injective**) iff

$$\forall\, x_1, x_2 \in X \;\; \big(f(x_1) = f(x_2) \;\rightarrow\; x_1 = x_2\big)$$

- Every element $x \in X$ is mapped to a unique element of Y
- Images are unique

$f : X \mapsto Y$ is **one-to-one** if there are no duplicates in the list

Every element of Y appears at most once

▷ No element of codomain is repeated

Properties of functions as lists

ICP 9-1 Suppose X and Y are two finite sets.

Let $f : X \mapsto Y$ be a one-to-one function.

What can we conclude about the cardinalities of X and Y?

$$|X| \textcircled{?} |Y|$$

Properties of functions as lists

A function $f : X \mapsto Y$ is **onto** (or **surjective**) iff
for every element $y \in Y$ there is an element $x \in X$ with $f(x) = y$

- Every element $y \in Y$ is assigned to some element of $x \in X$
- Every $y \in Y$ is the image of at least one $x \in X$

$f : X \mapsto Y$ is **onto** if every element in Y appears in the list

Every element of Y appear in the list at least once

▷ No element of codomain is missing

Properties of functions as lists

ICP 9-2 Suppose X and Y are two finite sets.

Let $f : X \mapsto Y$ be an onto function.

What can we conclude about the cardinalities of X and Y?

$$|X| \ ⓘ\ |Y|$$

Properties of functions as lists

A function $f : X \mapsto Y$ is **one-to-one correspondence** (or **bijective**) iff it is **both one-to-one** and **onto**

$f : X \mapsto Y$ is a bijection if every $y \in Y$ appears exactly once

Every $y \in Y$ appears $\underbrace{\text{at most once}}_{\text{one-to-one}}$ and $\underbrace{\text{at least once}}_{\text{onto}}$ in the list

If $f : X \mapsto Y$ is a bijection and X and Y are finite sets, then $|X| = |Y|$

Properties of functions as lists

Let $f : X \mapsto Y$ be represented as list

$f : X \mapsto Y$ is **one-to-one** if every $y \in Y$ appears at most once in the list

$f : X \mapsto Y$ is **onto** if every $y \in Y$ appears at least once in the list

$f : X \mapsto Y$ is **bijection** if every $y \in Y$ appears exactly once in the list

If $f : X \mapsto Y$ is a bijection and X and Y are finite sets, then $|X| = |Y|$

Properties of functions as lists

ICP 9-3

Given $f : A \rightarrow B$, and $|B| > |A|$. Explain why f cannot be a bijection.

ICP 9-4

Given $f : A \rightarrow B$, and $|B| < |A|$. Explain why f cannot be a bijection.

ICP 9-5

Given $f : A \rightarrow B$, and $|B| = |A|$. Does f have to be bijection?

Cardinality of finite sets

- Cardinality of a finite set X is the number of distinct elements in X
- X and Y have the same cardinality, if $|X| = |Y|$
- If $f : X \mapsto Y$ is a bijection and X and Y are finite sets, then $|X| = |Y|$

For finite sets X and Y, $|X| = |Y|$ iff there is a bijection $f : X \mapsto Y$

Proof of the if part

"if $|X| = |Y|$, then there is a bijection"

- Fix any ordering on X and Y
- $f(x_i) = y_i$ is a bijection

Cardinality of finite sets

For finite sets X and Y, $|X| = |Y|$ iff there is a bijection $f : X \mapsto Y$

$X = \{apple, 20, banana\}$

$Y = \{a, b, c\}$

apple	*c*
20	*b*
banana	*a*

$X = \{1, 2, 3, 4, 5\}$

$Y = \{a, b, c, d, e\}$

1	*a*
2	*b*
3	*c*
4	*d*
5	*e*

The power set

- Let A be a set such that $|A| = n$
- Impose any ordering on elements of A
- Represent subsets of A by n-bit strings (bit-vector representation)
- Each bit stands for whether the corresponding element is in the subset

ICP 9-6 How many n-bits strings are there?

ICP 9-7 How many subsets of the set A are there?

ICP 9-8 Design a bijection between the set of n-bits strings and the power set of A.

Cardinality of finite sets

For finite sets X and Y, $|X| = |Y|$ iff there is a bijection $f : X \mapsto Y$

$X = \{1, 2, 3, 4, 5, 6\}$
$Y = \{a, b, c, d, e\}$

In any $f : X \mapsto Y$

- some $y \in Y$ must repeat
- f cannot be one-to-one
- No bijection between X and Y
- so $|X| \neq |Y|$

$X = \{1, 2, 3, 4, 5, 6\}$
$Y = \{a, b, c, d, e, f, g\}$

In any $f : X \mapsto Y$

- some $y \in Y$ must be missing
- f cannot be onto
- No bijection between X and Y
- so $|X| \neq |Y|$