Proofs

- Proofs: Terminology and Rules of Inference
- Direct Proof
- Proof by Contrapositive
- Proof by Contradiction
- Proofs using Case Analysis

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Proof

An argument that convincingly demonstrates the truth of a statement

In mathematics,

A proof is a chain of logical deductions that demonstrates the truth of a proposition assuming the truth of some known axioms

- Direct proof to prove $P \to Q$
  - Assume P is true, with a chain of logical deductions conclude that Q is true
- Proof by contrapositive to prove $P \to Q$
  - Give a direct proof of $\neg Q \to \neg P$
- Proof by contradiction to prove P
  - Assume $\neg P$, with a chain of logical deductions draw a contradiction

Proofs using Case Analysis

Proof using case analysis is not a proof method as such

- Break up complicated proof into cases
- Deal with each case separately (applying either of the methods above)
- Must cover all cases

Proof using Case Analysis

Prove that The square of any real number is non-negative

Proof:

Any real number must satisfy one of these three cases

Case 1: $x = 0$, $x^2 = 0$ (non-negative)

Case 2: $x > 0$, $x^2 > 0$ (non-negative)

Case 3: $x < 0$, $x^2 > 0$ (non-negative)

We conclude that in all possible cases $x^2 \geq 0$ $\square$

Proof usingCase Analysis

ICP 8-10 Prove that the sum of two positive integers of the same parity (odd/even) is even.

Proof: Let x and y be the two positive integers (summands)

Case 1: Both x and y are even

$x = 2a \qquad y = 2b \quad$ for $a, b \in \mathbb{Z}$

$x + y = ?$

Case 2: Both x and y are odd

$x = 2a + 1 \qquad y = 2b + 1 \quad$ for $a, b \in \mathbb{Z}$

$x + y = ?$

Proof using Case Analysis

Prove that   Every even perfect square is divisible by 4

Proof:

Let x be an even integer and a perfect square

Suppose x is not divisible by 4, then x must satisfy one of three cases

Case 1: $x = 4k + 1, \implies x$ is odd

$\triangleright$ a contradiction to assumption that x is even

Case 2: $x = 4k + 2, \implies x = y^2$ is even, y is even, $y = 2a$

So $x = (2a)^2 = 4a^2$, hence 4 divides x

$\triangleright$ a contradiction to assumption that x is not divisible by 4

Case 3: $x = 4k + 3 \implies x$ is odd

$\triangleright$ a contradiction to assumption that x is even

$\square$

Proofs by Case Analysis

ICP 8-11 Prove that if x and y are real numbers, then $\max\{x, y\} + \min\{x, y\} = x + y$.

Proof:

Case 1: $x > y \implies \max\{x, y\} = x \quad \min\{x, y\} = y$

Case 2: $x < y \implies \max\{x, y\} = y \quad \min\{x, y\} = x$

Case 3: $x = y \implies \max\{x, y\} = x \quad \min\{x, y\} = x$

Proof using Case Analysis

Given any two persons they are either friends or they are strangers

Theorem

Every collection of 6 people contains a group of 3 friends or a group of 3 strangers

Let $\{a, b, c, d, e, f\}$ be an arbitrary collection of 6 people

Proof using Case Analysis

Theorem

Every collection of 6 people contains a group of 3 friends or a group of 3 strangers

Consider a (fixed) person a

Case 1: Among the remaining there are ≥ 3 people who are all friends with a

Case 2: Among the remaining there are ≥ 3 people who are all strangers to a

One of these two cases must happen

Proof using Case Analysis

Theorem

Every collection of 6 people contains a group of 3 friends or a group of 3 strangers

Case 1: Among the remaining there are ≥ 3 people who are all friends with a

Case 1.1:
Among the ≥ 3 friends of a there are two who are friends with each other

Case 1.2:
All the ≥ 3 friends of a are strangers to each other

All subcases covered

Proof using Case Analysis

Theorem

Every collection of 6 people contains a group of 3 friends or a group of 3 strangers

Case 2: Among the remaining there are ≥ 3 people who are all strangers to a

Case 2.1:
Among the ≥ 3 strangers to a there are two who are strangers to each other

Case 2.2: All the ≥ 3 strangers to a are friends with each other

All subcases covered

Proofs by Case Analysis

ICP 8-12 Prove that if x and y are real numbers, then $|x| + |y| \geq |x + y|$.

Proof: Without loss of generality assume $x \geq y$ $\triangleright$ if not rename them

Case 1: $y \geq 0 \implies x > 0$

Case 2: $y < 0 \quad x \geq 0$

Case 3: $x < 0 \implies y < 0$