Proofs

- Proofs: Terminology and Rules of Inference
- Direct Proof
- Proof by Contrapositive
- Proof by Contradiction
- Proofs using Case Analysis

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Proof

An argument that convincingly demonstrates the truth of a statement

In mathematics,

A proof is a chain of logical deductions that demonstrates the truth of a proposition assuming the truth of some known axioms

- Direct proof to prove $P \to Q$
  - Assume P is true, with a chain of logical deductions conclude that Q is true

Proof by Contrapositive

Recall that an implication is equivalent to it's contrapositive

$$P \to Q \;\equiv\; \neg Q \to \neg P$$

To prove $P \to Q$,

apply the direct proof method to it's contrapositive $(\neg Q \to \neg P)$

Just a restatement of the given statement rather than a proof method

Proof by Contrapositive

Prove that $\quad$ If $\underbrace{a + b > 7}_{P}$, then $\underbrace{a > 3 \text{ or } b > 4}_{Q}$

Contrapositive

Prove that $\quad$ If $\underbrace{\neg(a > 3 \text{ or } b > 4)}_{\neg Q}$, then $\underbrace{\neg(a + b > 7)}_{\neg P}$

Proof: Assume $\neg Q$ is true

$\neg((a > 3) \vee (b > 4)) \equiv (a \leq 3) \wedge (b \leq 4)$ $\quad \triangleright$ DeMorgan Law

Adding the two inequalities, we get

$a + b \leq 3 + 4 = 7$, and equivalently

$\neg(a + b > 7) \equiv \neg P$

We conclude that $\neg Q \to \neg P$ and hence $P \to Q$ is true $\quad \square$

Proof by Contrapositive

Prove that   If $x^5 < 0$, then $x < 0$

Contrapositive

Prove that   If $x \geq 0$, then $x^5 \geq 0$

Proof: Assume $x \geq 0$

If $x = 0$, then $x^5 = 0 \quad \implies \quad x^5 \geq 0$

If $x > 0$, then

since the product of five positive numbers is positive, hence

$x^5 > 0 \geq 0$

We conclude that $\neg Q \to \neg P$ and hence $P \to Q$ is true $\square$

ICP 8-4 Prove that

For all positive integers n, if 5 does not divide n^2, then 5 does not divide n

Contrapositive:

if n is a multiple of 5, then n^2 is a multiple of 5

Proof: $n = 5k \implies n^2 = (5k)^2 = 5(5k^2)$

Proof by Contrapositive

Square root of an irrational number is an irrational number

In implication form the statement is

If r is irrational, then $\sqrt{r}$ is also irrational

Contrapositive

If $\sqrt{r}$ is rational, then r is rational

Proof by Contrapositive

If $\sqrt{r}$ is rational, then r is rational

Proof:

Let $\sqrt{r} = {}^{x}/_{y}$ where x and y are integers and $y \neq 0$

Squaring both sides, we get

$$r = \frac{x^2}{y^2},$$

x^2 and y^2 are integers, and $y^2 \neq 0$

hence r is a rational number

Proofs by Contrapositive

ICP 8-5 Prove that for all $n \in \mathbb{N}$, if $n^2 + 3$ is an odd integer, then n is an even integer.

Contrapositive:

if n is odd, then $n^2 + 3$ is even

Proof:

$$n = 2k + 1 \implies n^2 + 3 = 4k^2 + 4k + 1 + 3 = 2(2k^2 + 2k + 2)$$