Proofs

- Proofs: Terminology and Rules of Inference
- Direct Proof
- Proof by Contrapositive
- Proof by Contradiction
- Proofs using Case Analysis

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Proof

An argument that convincingly demonstrates the truth of a statement

In mathematics,

A proof is a chain of logical deductions that demonstrates the truth of a proposition assuming the truth of some known axioms

Direct Proofs

Direct Proof: used to prove statement of the form $P \to Q$

1. Assume that P is true
2. With a chain of logical deductions conclude that Q is true

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

When P is false, $P \to Q$ is already true irrespective of value of Q

The only case when $P \to Q$ is false, is when $P = T$ and $Q = F$

Hence our goal is to rule out that possibility

Direct Proofs

Using a direct proof method, we will prove the following statement

Definition

An integer y is even if there exists an integer x such that $y = 2x$

Theorem

The product of two even integers is even

Restatement as an implication

Theorem

If x and y are even integers, then xy is even

Direct Proofs

Theorem

If $\underbrace{x \text{ and } y \text{ are even integers}}_{P}$, then $\underbrace{xy \text{ is even}}_{Q}$

Proof: Assume that P is true

Let $x = 2a$ and $y = 2b$, then

$$\begin{aligned} xy &= (2a)(2b) \\ &= 4(ab) \\ &= 2(2ab) \end{aligned}$$

Since $2ab$ is an integer, by definition xy is an even integer so Q is true

We conclude that $P \to Q$ is true □

Direct Proofs

Theorem

If $\underbrace{a > 3 \text{ and } b > 4}_{P}$, *then* $\underbrace{a + b > 7}_{Q}$

Proof: Assume that P is true

Let $a > 3$ and $b > 4$,

Adding the two inequalities, we get

$$a + b \; > \; 3 + 4 \; = \; 7$$

Hence Q is also true

We conclude that $P \to Q$ is true $\square$

Direct Proofs

ICP 8-1 Prove or disprove that the product of two consecutive integers is an even integer.

In any two consecutive integers

- one of them will be even $(2m)$
- and the other will be odd $(2m \pm 1)$

$$2m \times (2m \pm 1) \;=\; 2(2m^2 \pm m)$$

Hence the product is even □

Direct Proofs

Theorem

The sum of two rational numbers is rational

Restating as an implication

Theorem

If p and q are rational numbers, then $p + q$ is rational

Direct Proofs

Theorem

If p and q are rational numbers, then $p + q$ is rational

Proof:

Let $p = {}^1\!/_3$ and $q = {}^3\!/_4$

$$p + q \quad = \quad {}^1\!/_3 + {}^3\!/_4 \quad = \quad {}^{13}\!/_{12}$$

which is a rational number!

Not a Proof: This only shows one example, (like a program that works for only one input)

Direct Proofs

Theorem

If p and q are rational numbers, then $p + q$ is rational

Proof:

Let $p = {}^{a}/_{b}$ for some integers a and $b \neq 0$

and let $q = {}^{a}/_{b}$ for some integers a and $b \neq 0$

then $p + q = 2({}^{a}/_{b}) = {}^{2a}/_{b}$

Since $2a$ and b are integers, $p + q$ is a rational number!

Not a Proof: This only works for cases when $p = q$

Direct Proofs

Theorem

If p and q are rational numbers, then $p + q$ is rational

Proof:

Let $p = {}^{a}/_{b}$ for some integers a and $b \neq 0$

and let $q = {}^{c}/_{d}$ for some integers c and $d \neq 0$

- rational numbers are fractions
- p and q are fractions and sum of fractions is a fraction
- $p + q$ is a fraction, thus $p + q$ is a rational number

Not a Proof: wrong definition of rational numbers

all rationals are fractions but not all fractions are rational numbers, ${}^{\pi}/_{1}$

Direct Proofs

Theorem

If p and q are rational numbers, then $p + q$ is rational

Proof:

Let $p = a/b$ for some integers a and b, $b \neq 0$
and let $q = c/d$ for some integers c and d, $d \neq 0$

then $p + q \; = \; a/b + c/d \; = \; (ad+bc)/bd$

$(ad + bc)$ and bd are integers and $bd \neq 0$

Thus, $p + q$ is a rational number

$\square$

How many bad proofs are there!

Direct Proofs

ICP 8-2 Prove or disprove that the product of two odd integers is an odd integer.

Given two odd integers $2m+1$ and $2n+1$:

$$(2m+1) \times (2n+1) \; = \; 4mn + 2m + 2n + 1 \; = \; 2(2mn + m + n) + 1$$

Hence the product is an odd integer $\square$

Direct Proofs

ICP 8-3 Prove or disprove that the product of any five consecutive positive integers is divisible by 120.

Given five consecutive integers, n, $n+1$, $n+2$, $n+3$ and $n+4$:

- Exactly one of them is divisible by 5 $\implies$ their product is divisible by 5
- At least one of them is divisible by 4 $\implies$ their product is divisible by 4
- At least one of them is divisible by 3 $\implies$ their product is divisible by 3
- At least one of them is divisible by 2 $\implies$ their product is divisible by 2

Hence the product is divisible by $5 \times 4 \times 3 \times 2 = 120$ $\square$