Sequences and Sums

- Sequences and Progressions
- Summation and its linearity
- Evaluating Sums
- Evaluating Sums - Proofs without words
- Geometric Sums

Imdad ullah Khan

Evaluating Sums

- Given a summation find a *closed form* formula to evaluate it
- A formula to output value of the sum using the values of limits only

$$\sum_{i=1}^{n} i = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

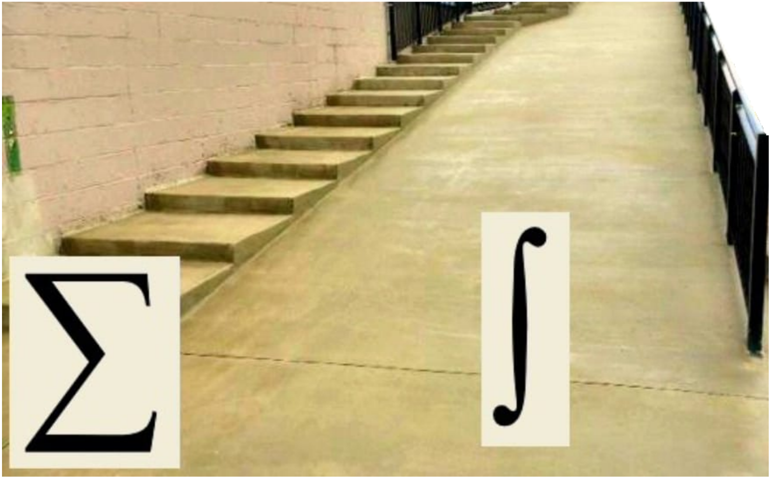
$$\sum_{i=1}^{n} 2i = 2 + 4 + \ldots + 2n = n(n+1)$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{i=1}^{n} 2i - 1 = 1 + 3 + \ldots + 2n - 1 = n^2$$

Evaluating Sums



Geometric Progressions

A geometric progression is a sequence of the form

$$a, ar, ar^2, \dots, ar^i, ar^{i+1} \dots$$

where a and r are real numbers

$$\frac{ar^{i+1}}{ar^i} = r$$

- the ratio of consecutive terms is called the common ratio
- a is called the initial term
- The next term is obtained by multiplying the previous term with r

Finite Geometric Sums

Evaluating finite geometric sum

$$S_n = \sum_{i=0}^{n-1} ar^i$$

Add one more term

$$S_n + ar^n = \sum_{i=0}^{n-1} ar^i + ar^n = ar^0 + \sum_{i=0}^{n-1} ar^{i+1}$$

$$= ar^0 + r\underbrace{\sum_{i=0}^{n-1} ar^i}_{S_n} = ar^0 + rS_n = a + rS_n$$

$$S_n + ar^n = a + rS_n \implies S_n = \frac{a - ar^n}{1 - r} \quad \text{when } r \neq 1$$

Finite Geometric Sums

$$S_n = \sum_{i=0}^{n-1} ar^i \quad \Longrightarrow \quad S_n = \frac{a - ar^n}{1 - r} \quad \text{when } r \neq 1$$

Some authors refer to this method as the **perturbation method**.

Generally, it works as follows

1. Denote an unknown sum as $S_n = \sum_{i=0}^{n} a_i$
2. $S_{n+1} = S_n + a_{n+1} = a_0 + \sum_{i=1}^{n+1} a_i$ ▷ separate last and first term
3. $S_{n+1} = S_n + a_{n+1} = a_0 + \sum_{j=0}^{n} a_{j+1}$ ▷ change of variable
4. Try to express $\sum_{j=0}^{n} a_{j+1}$ in terms of S_n
5. If can be expressed as multiple of S_n, solve for S_n to get a closed form

Finite Geometric Sums

ICP 7-11 Use the perturbation method to find a closed form of

$$\sum_{i=0}^{n} i\, r^{i}$$

Infinite Geometric Series

Evaluating infinite geometric sum

$$S = \sum_{i=0}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=0}^{n} a_i$$

$$S = \sum_{i=0}^{\infty} ar^i = \lim_{n\to\infty} \sum_{i=0}^{n-1} ar^i = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a - ar^n}{1-r}$$

$$S = \lim_{n\to\infty} \frac{a - ar^n}{1-r} = \begin{cases} \frac{a}{1-r} & if \;\; |r| < 1 \\ \infty & else \end{cases}$$

Infinite Geometric Series

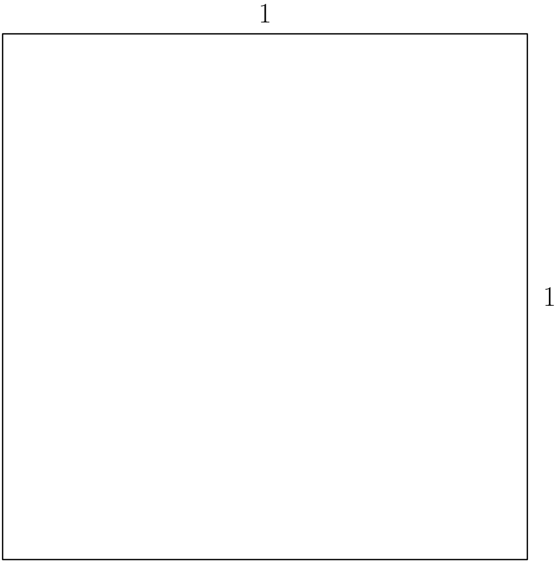
$$S = \sum_{i=0}^{\infty} ar^i = \begin{cases} \dfrac{a}{1-r} & \textit{if} \;\; |r| < 1 \\ \infty & \textit{else} \end{cases}$$

Evaluate $\sum_{i=0}^{\infty} (1/2)^i$ $\quad (a = 1, r = \frac{1}{2})$

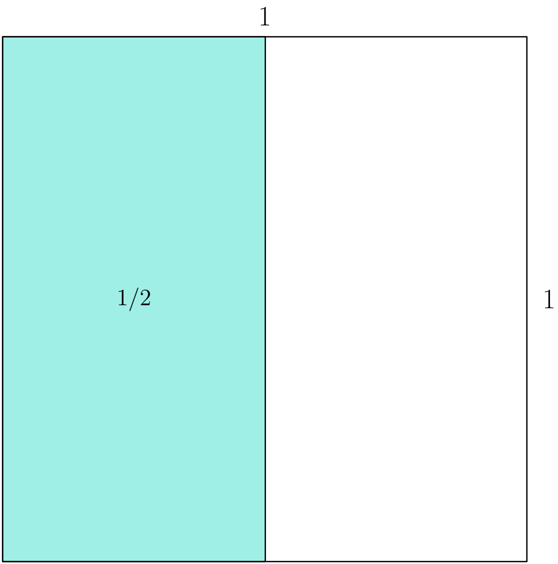
$$\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots = 2$$

$$\sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = 1$$

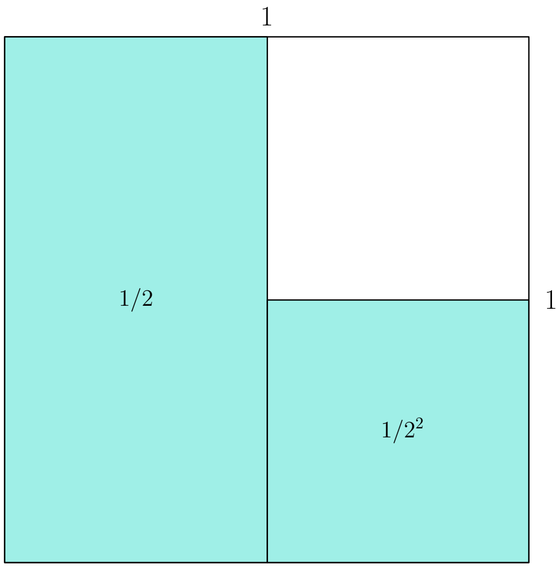
Infinite Geometric Series: Geometric Approach



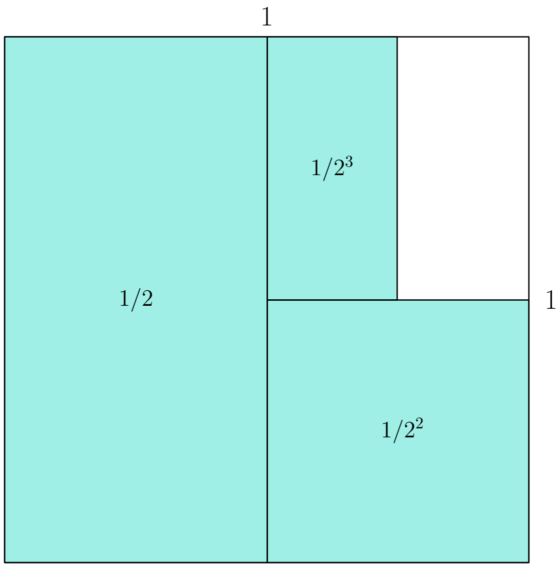
Geometric Sum: Geometric Approach



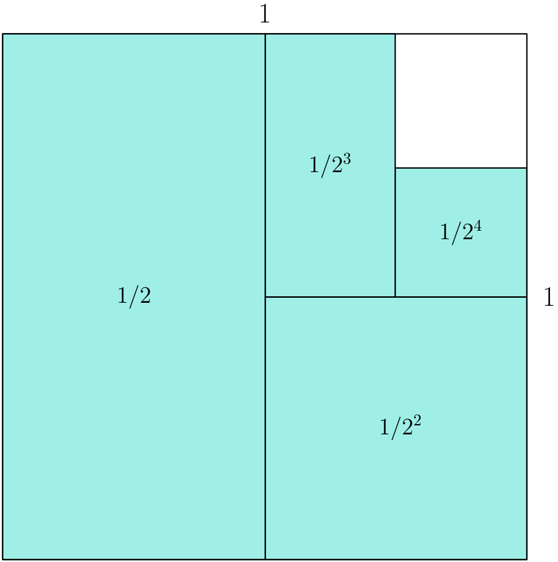
Geometric Sum: Geometric Approach



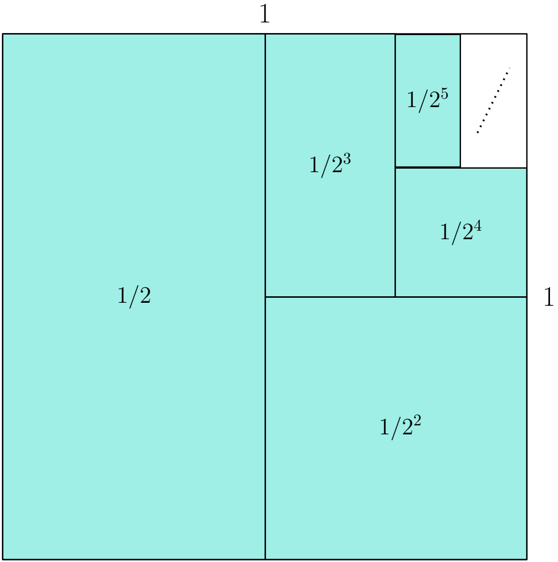
Geometric Sum: Geometric Approach



Geometric Sum: Geometric Approach



Geometric Sum: Geometric Approach



Finite Geometric Sum from Infinite

$$S_n = \sum_{i=0}^{n-1} ar^i = \frac{a - ar^n}{1 - r} \quad \text{when } r \neq 1$$

$$S = \sum_{i=0}^{\infty} ar^i = \begin{cases} \dfrac{a}{1-r} & \text{if } |r| < 1 \\ \infty & \text{else} \end{cases}$$

An alternative way to evaluate S_n

$$S = \underbrace{a + ar + ar^2 + \ldots + ar^{n-1}}_{S_n} + ar^n + ar^{n+1} + ar^{n+2} + \ldots$$

$$S = S_n + ar^n + ar^{n+1} + ar^{n+2} + ar^{n+3} + \ldots$$

Finite Geometric Sum from Infinite

An alternative way to evaluate S_n

$$S = \underbrace{a + ar + ar^2 + \ldots + ar^{n-1}}_{S_n} + ar^n + ar^{n+1} + ar^{n+2} + \ldots$$

$$= S_n + ar^n + ar^{n+1} + ar^{n+2} + ar^{n+3} + \ldots$$

$$= S_n + r^n \left(a + ar + ar^2 + ar^3 + \ldots\right)$$

$$S = S_n + r^n S$$

$$\implies S_n = S\left(1 - r^n\right)$$

$$\implies S_n = \frac{a\left(1 - r^n\right)}{1 - r}$$

Evaluating Sums

Evaluate the following sums in terms of appropriate variables

ICP 7-12 $\displaystyle\sum_{i=0}^{\infty} 2^i/3^{i+1} = \frac{1}{3}\sum_{i=0}^{\infty} 2^i/3^i = \frac{1}{3}\sum_{i=0}^{\infty} (2/3)^i$

▷ Infinite geometric sum $a = 1$, $r = 2/3$

ICP 7-13 $\displaystyle\sum_{i=0}^{n} 2^i/3^{i+1}$

ICP 7-14 $\displaystyle\sum_{i=3}^{\infty} 2^{i+1}/3^i$

ICP 7-15 $\displaystyle\sum_{i=2}^{n} 2^{i+1}/3^i$