Sequences and Sums

- Sequences and Progressions
- Summation and its linearity
- Evaluating Sums
- Evaluating Sums - Proofs without words
- Geometric Sums

Imdad ullah Khan

Evaluating Sums

- Given a summation find a *closed form* formula to evaluate it

$$\sum_{i=1}^{n} i = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} 2i = 2 + 4 + \ldots + 2n = n(n+1)$$

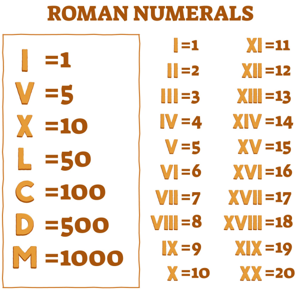
$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

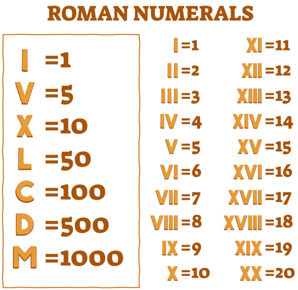
$$\sum_{i=1}^{n} 2i - 1 = 1 + 3 + \ldots + 2n - 1 = n^2$$

We will derive some of these sums using proofs without words

Evaluating Sums

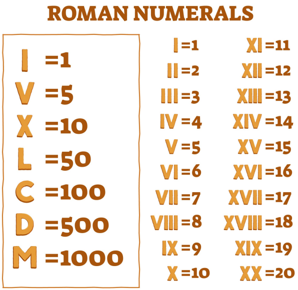


Evaluating Sums



$$MDCIX + MCVI \;\; = ?$$

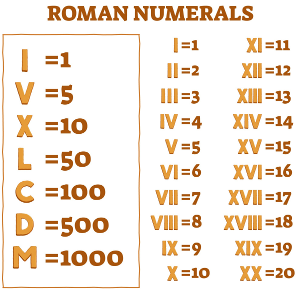
Evaluating Sums



$$\begin{array}{rcccc} & & & 1 & \\ & 1 & 6 & 0 & 9 \\ + & 1 & 1 & 0 & 6 \\ \hline & 2 & 7 & 1 & 5 \end{array}$$

$$MDCIX + MCVI \;\; = ?$$

Evaluating Sums



$$MDCIX + MCVI \ = ?$$

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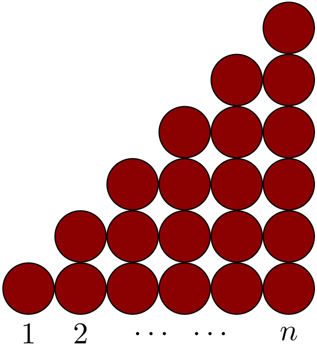
The genius of positional number system

Muhammad ibn Musa al-Khwarizmi

Evaluating Sums

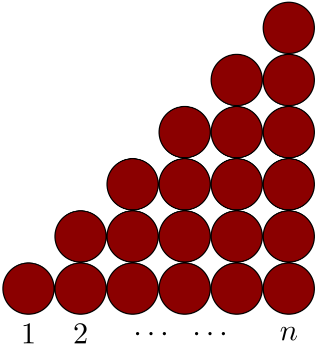
Evaluate $T_n := \sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n - 2 + n - 1 + n$



Evaluating Sums

$$\text{Evaluate} \quad T_n := \sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n - 2 + n - 1 + n$$

T_n is the number of red balls

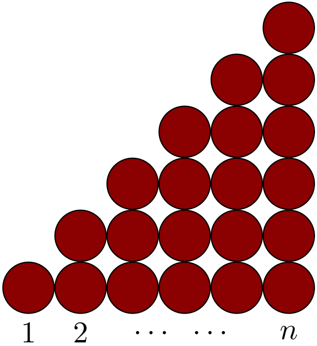


Evaluating Sums

$$\text{Evaluate} \quad T_n := \sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n - 2 + n - 1 + n$$

T_n is the number of red balls

T_n : nth triangular number



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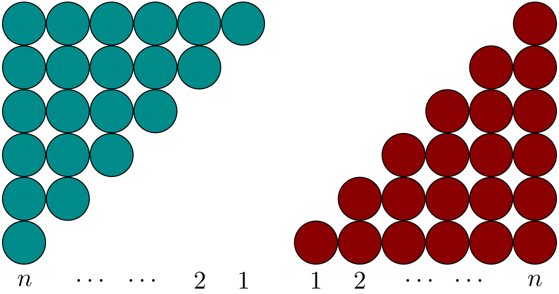
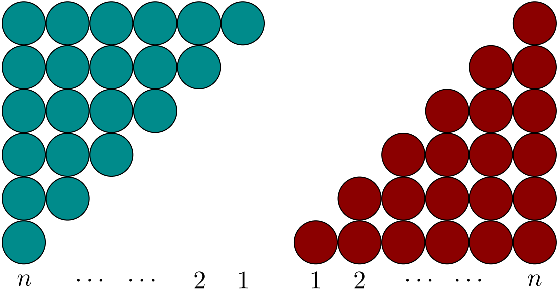


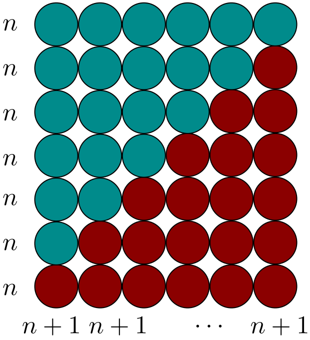
Evaluating Sums

Evaluate $T_n := \sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n - 2 + n - 1 + n$



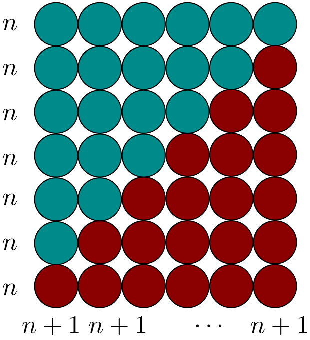
Double the number of balls, $2T_n$

Evaluating Sums



Evaluating Sums

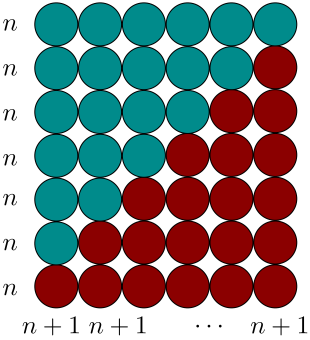
rectangular grid $n \times (n+1)$



Evaluating Sums

rectangular grid $n \times (n+1)$

number of balls $= 2T_n$

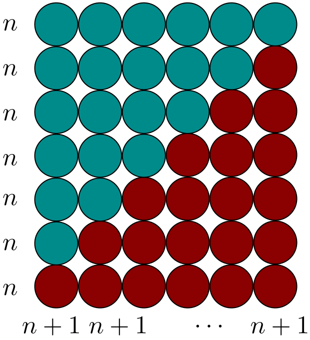


Evaluating Sums

rectangular grid $n \times (n+1)$

number of balls $= 2T_n$

$\Longrightarrow\ 2T_n \ = \ n(n+1)$



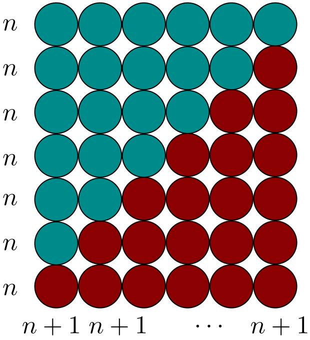
Evaluating Sums

rectangular grid $n \times (n+1)$

number of balls $= 2T_n$

$$\implies 2T_n = n(n+1)$$

$$T_n = \frac{n(n+1)}{2}$$



Triangular number

$$T_n = 1 + 2 + \ldots + n - 1 + n$$

Triangular number

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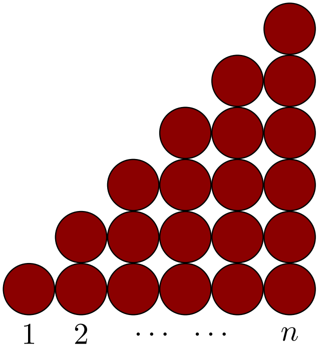
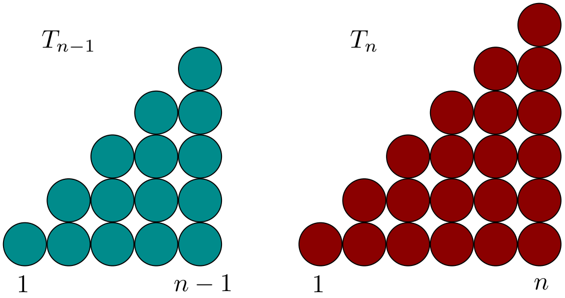
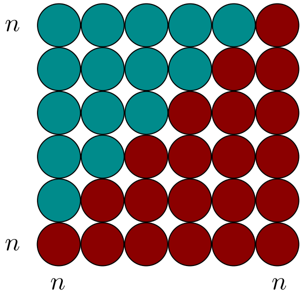


Square Numbers

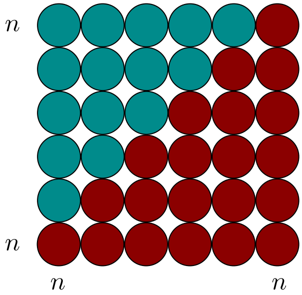


Square Numbers



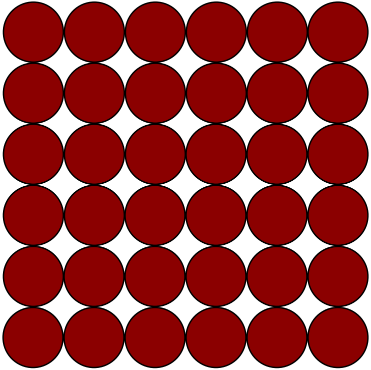
Square Numbers

$$n^2 = T_n + T_{n-1}$$



Sum of Odd Numbers

Evaluate $$O_n := \sum_{i=1}^{n}(2i-1) \;=\; 1+3+5+\ldots+(2n-1)$$



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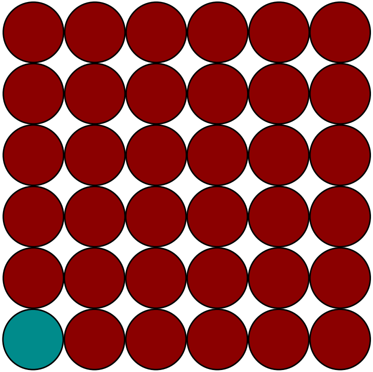

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Sum of Odd Numbers

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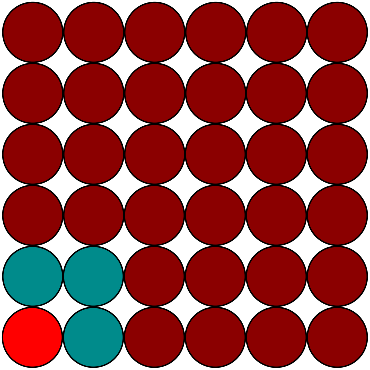
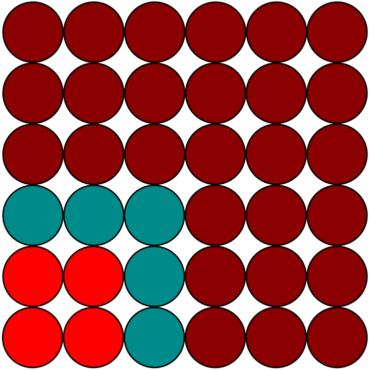


$1+3$

Sum of Odd Numbers

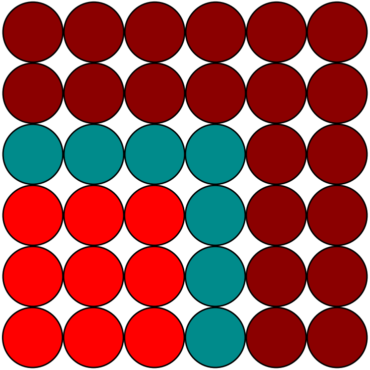
Evaluate $$O_n := \sum_{i=1}^{n}(2i-1) \;=\; 1+3+5+\ldots+(2n-1)$$



$1+3+5$

Sum of Odd Numbers

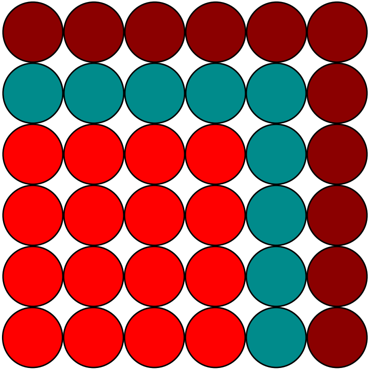
Evaluate $$O_n := \sum_{i=1}^{n}(2i-1) = 1+3+5+\ldots+(2n-1)$$



$$1+3+5+7$$

Sum of Odd Numbers

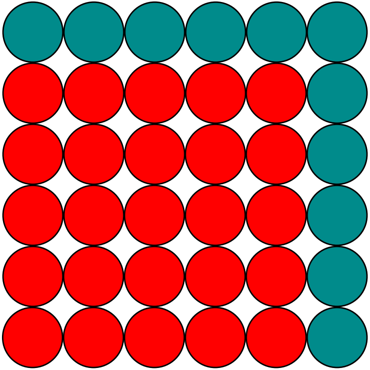
Evaluate $$O_n := \sum_{i=1}^{n}(2i-1) \; = \; 1+3+5+\ldots+(2n-1)$$



$$1+3+5+7+9$$

Sum of Odd Numbers

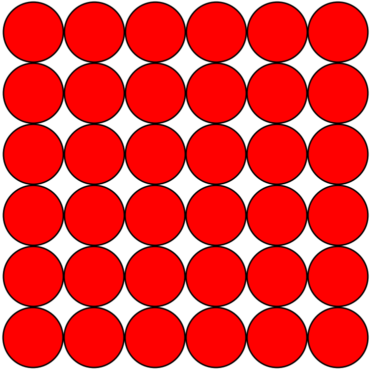
Evaluate $$O_n := \sum_{i=1}^{n}(2i-1) = 1+3+5+\ldots+(2n-1)$$



$$1+3+5+7+9+11$$

Sum of Odd Numbers

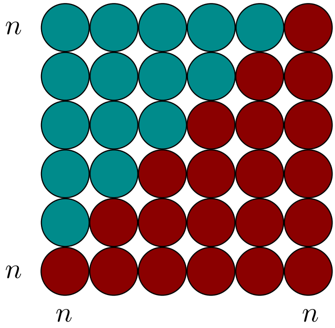
Evaluate $$O_n := \sum_{i=1}^{n}(2i-1) \;=\; 1+3+5+\ldots+(2n-1)$$



$$1+3+5+7+9+11=36$$

Square Numbers

$n^2 = T_n + T_{n-1}$



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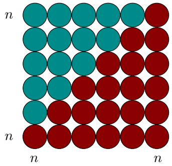

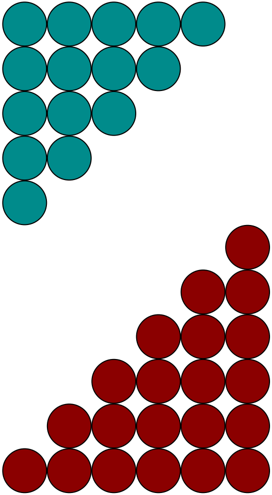

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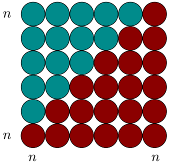

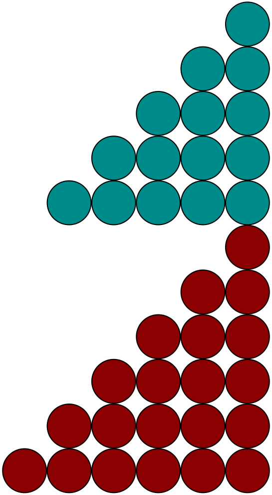

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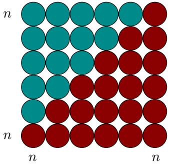
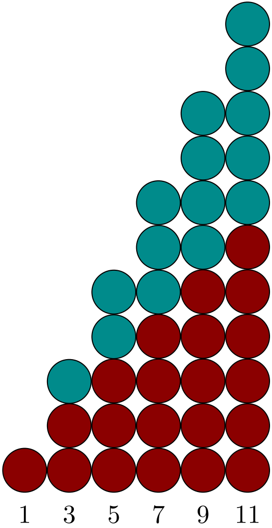




Sum of Squares and Cubes

Evaluate sum of squares and sum of cubes of first n positive integers

$$S_n := \sum_{i=1}^{n} i^2 := 1^2 + 2^2 + 3^2 + \ldots + n^2$$

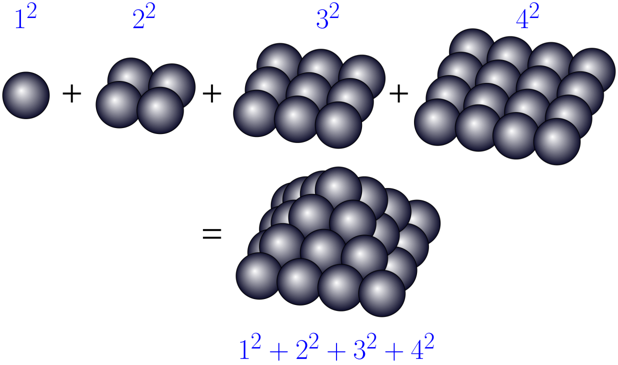
and

$$C_n := \sum_{i=1}^{n} i^3 := 1^3 + 2^3 + 3^3 + \ldots + n^3$$

Sum of Squares

Evaluate sum of squares and of first n positive integers

$$S_n := \sum_{i=1}^{n} i^2 := 1^2 + 2^2 + 3^2 + \ldots + n^2$$



Sum of Squares: Application

Evaluate sum of squares and of first n positive integers

$$S_n := \sum_{i=1}^{n} i^2 := 1^2 + 2^2 + 3^2 + \ldots + n^2$$

$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + \vdots$

Sum of Squares: Application

Evaluate sum of squares and of first n positive integers

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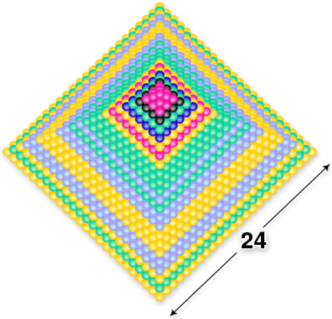

Sum of Squares: Application

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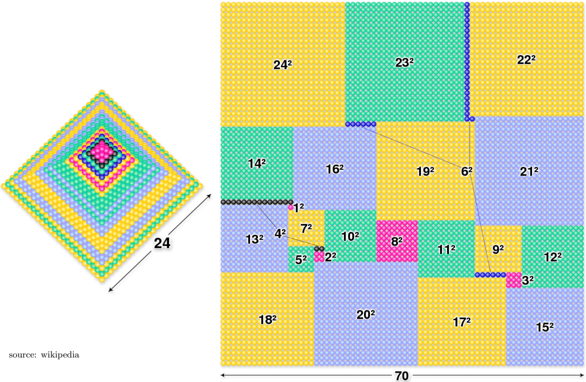


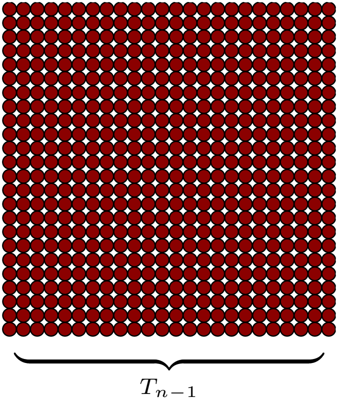
source: wikipedia

Sum of Cubes

$$\text{Evaluate} \quad C_n := 1^3 + 2^3 + 3^3 + \ldots + n^3$$

Sum of Cubes

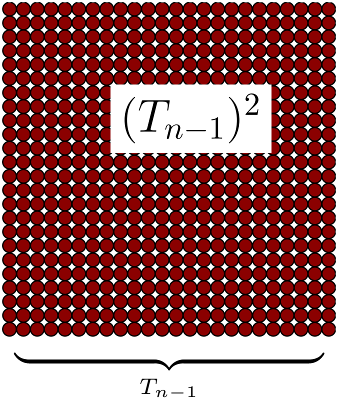
Evaluate $C_n := 1^3 + 2^3 + 3^3 + \ldots + n^3$



Sum of Cubes

Evaluate $C_n := 1^3 + 2^3 + 3^3 + \ldots + n^3$

Area of the square: $(T_{n-1})^2$

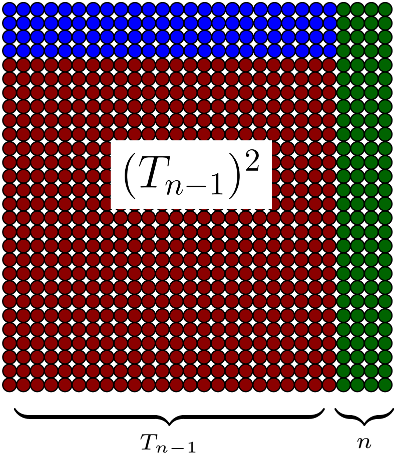


Sum of Cubes

Evaluate $C_n := 1^3 + 2^3 + 3^3 + \ldots + n^3$

Area of old square: $(T_{n-1})^2$

Add n rows and n columns



Sum of Cubes

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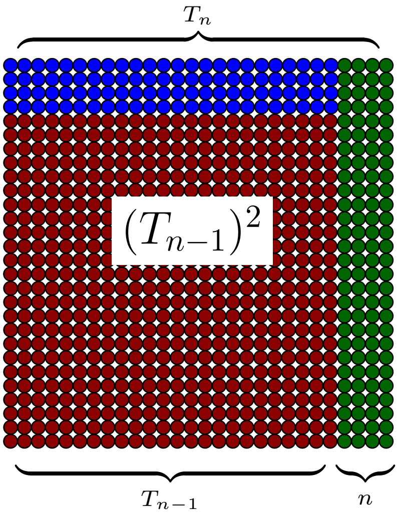


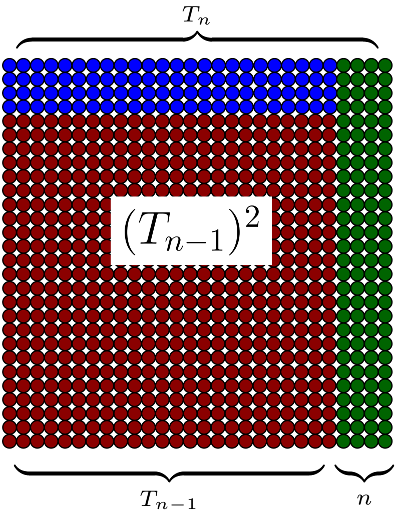
Sum of Cubes

Evaluate $C_n := 1^3 + 2^3 + 3^3 + \ldots + n^3$

Area of old square: $(T_{n-1})^2$

Add n rows and n columns

Area of the new square: $(T_n)^2$



Sum of Cubes

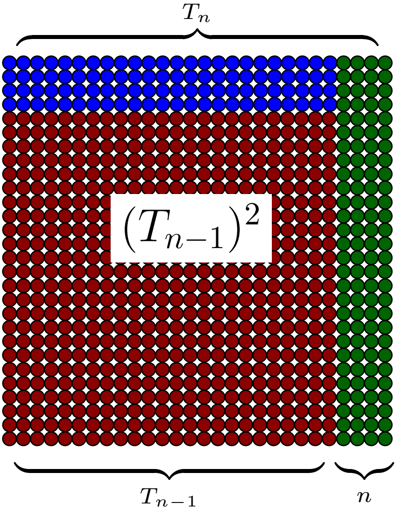
Evaluate $C_n := 1^3 + 2^3 + 3^3 + \ldots + n^3$

Area of old square: $(T_{n-1})^2$

Add n rows and n columns

Area of the new square: $(T_n)^2$

Area of new region:



Sum of Cubes

Evaluate $C_n := 1^3 + 2^3 + 3^3 + \ldots + n^3$

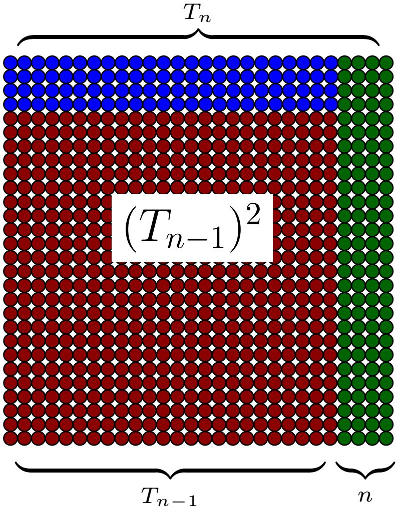
Area of old square: $(T_{n-1})^2$

Add n rows and n columns

Area of the new square: $(T_n)^2$

Area of new region:

$$= (T_n)^2 - (T_{n-1})^2$$



Sum of Cubes

Evaluate $C_n := 1^3 + 2^3 + 3^3 + \ldots + n^3$

Area of old square: $(T_{n-1})^2$

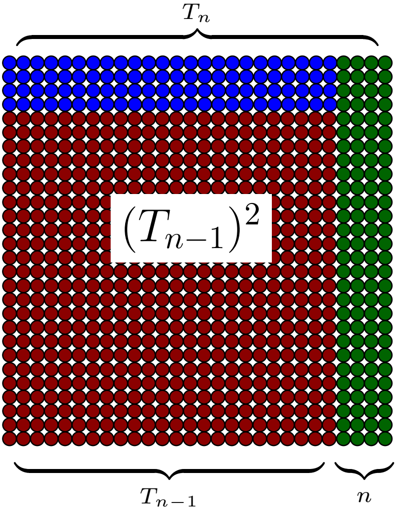
Add n rows and n columns

Area of the new square: $(T_n)^2$

Area of new region:

$$= (T_n)^2 - (T_{n-1})^2$$

$$= n \cdot T_{n-1} + n \cdot T_n$$



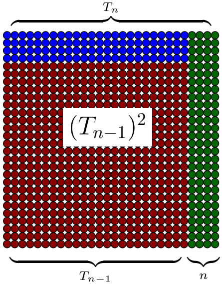
Sum of Cubes

$$\text{Evaluate} \quad C_n := 1^3 + 2^3 + 3^3 + \ldots + n^3$$

Area of old square: $(T_{n-1})^2$

Add n rows and n columns

Area of the new square: $(T_n)^2$



Area of new region:

$$= (T_n)^2 - (T_{n-1})^2$$

$$= n \cdot T_{n-1} + n \cdot T_n$$

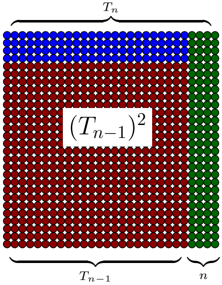
Sum of Cubes

$$\text{Evaluate} \quad C_n := 1^3 + 2^3 + 3^3 + \ldots + n^3$$

Area of old square: $(T_{n-1})^2$

Add n rows and n columns

Area of the new square: $(T_n)^2$



Area of new region:

$$= (T_n)^2 - (T_{n-1})^2$$

$$= n \cdot T_{n-1} + n \cdot T_n$$

$$= n(T_{n-1} + T_n)$$

Sum of Cubes

$$\text{Evaluate} \quad C_n := 1^3 + 2^3 + 3^3 + \ldots + n^3$$

Area of old square: $(T_{n-1})^2$

Add n rows and n columns

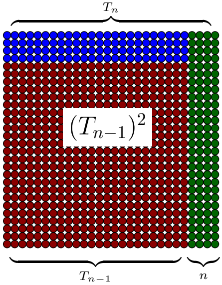
Area of the new square: $(T_n)^2$

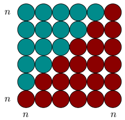
Area of new region:

$$= (T_n)^2 - (T_{n-1})^2$$

$$= n \cdot T_{n-1} + n \cdot T_n$$

$$= n(T_{n-1} + T_n)$$





$n^2 = T_n + T_{n-1}$

Sum of Cubes

$$\text{Evaluate} \quad C_n := 1^3 + 2^3 + 3^3 + \ldots + n^3$$

Area of old square: $(T_{n-1})^2$

Add n rows and n columns

Area of the new square: $(T_n)^2$

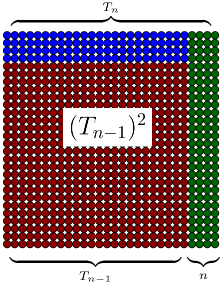
Area of new region:

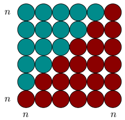
$$= (T_n)^2 - (T_{n-1})^2$$

$$= n \cdot T_{n-1} + n \cdot T_n$$

$$= n(T_{n-1} + T_n)$$

$$= n(n^2) = n^3$$





$n^2 = T_n + T_{n-1}$

Sum of Cubes

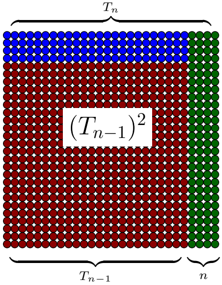
$$\text{Evaluate} \quad C_n := 1^3 + 2^3 + 3^3 + \ldots + n^3$$

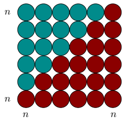
Area of old square: $(T_{n-1})^2$

Area of the new square: $(T_n)^2$

Area of new region:

$$= (T_n)^2 - (T_{n-1})^2 = n^3$$





$$n^2 = T_n + T_{n-1}$$

Sum of Cubes

$$\text{Evaluate} \quad C_n := 1^3 + 2^3 + 3^3 + \ldots + n^3$$

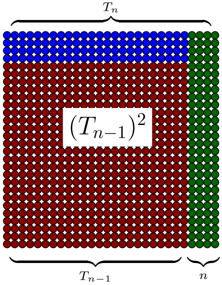
Area of old square: $(T_{n-1})^2$

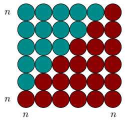
Area of the new square: $(T_n)^2$

Area of new region:

$$= (T_n)^2 - (T_{n-1})^2 = n^3$$

$$\implies (T_n)^2 = (T_{n-1})^2 + n^3$$





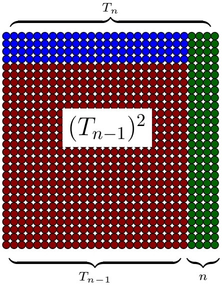
$$n^2 = T_n + T_{n-1}$$

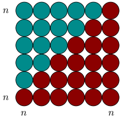
Sum of Cubes

Evaluate $C_n := 1^3 + 2^3 + 3^3 + \ldots + n^3$

$(T_n)^2 = (T_{n-1})^2 + n^3$

$= (T_{n-1})^2 + n^3$





$n^2 = T_n + T_{n-1}$

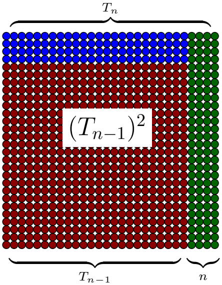
Sum of Cubes

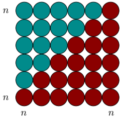
$$\text{Evaluate} \quad C_n := 1^3 + 2^3 + 3^3 + \ldots + n^3$$

$(T_n)^2 = (T_{n-1})^2 + n^3$

$$= (T_{n-1})^2 + n^3$$

$$= (T_{n-2})^2 + (n-1)^3 + n^3$$





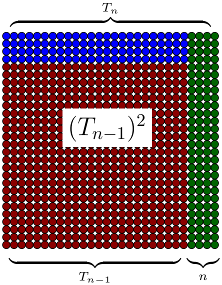
$n^2 = T_n + T_{n-1}$

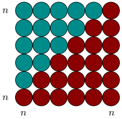
Sum of Cubes

$$\text{Evaluate} \quad C_n := 1^3 + 2^3 + 3^3 + \ldots + n^3$$

$(T_n)^2 = (T_{n-1})^2 + n^3$

$$\begin{aligned} &= (T_{n-1})^2 + n^3 \\ &= (T_{n-2})^2 + (n-1)^3 + n^3 \\ &= (T_{n-3})^2 + (n-2)^3 + (n-1)^3 + n^3 \end{aligned}$$





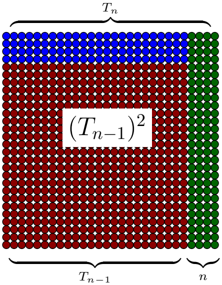
$n^2 = T_n + T_{n-1}$

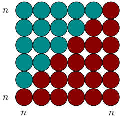
Sum of Cubes

$$\text{Evaluate} \quad C_n := 1^3 + 2^3 + 3^3 + \ldots + n^3$$

$$(T_n)^2 = (T_{n-1})^2 + n^3$$

$$\begin{aligned} &= (T_{n-1})^2 + n^3 \\ &= (T_{n-2})^2 + (n-1)^3 + n^3 \\ &= (T_{n-3})^2 + (n-2)^3 + (n-1)^3 + n^3 \\ &= \vdots \end{aligned}$$





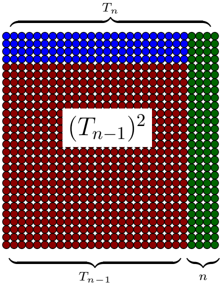
$$n^2 = T_n + T_{n-1}$$

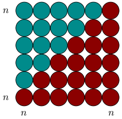
Sum of Cubes

Evaluate $C_n := 1^3 + 2^3 + 3^3 + \ldots + n^3$

$(T_n)^2 = (T_{n-1})^2 + n^3$

$$\begin{aligned} &= (T_{n-1})^2 + n^3 \\ &= (T_{n-2})^2 + (n-1)^3 + n^3 \\ &= (T_{n-3})^2 + (n-2)^3 + (n-1)^3 + n^3 \\ &= \vdots \\ &= 1^3 + 2^3 + \ldots (n-1)^3 + n^3 \end{aligned}$$





$n^2 = T_n + T_{n-1}$

Sum of Cubes

Evaluate

$$C_n := \sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3$$

$$\begin{aligned} 1^3 + 2^3 + \ldots (n-1)^3 + n^3 &= (T_n)^2 \\ &= \left(\frac{n(n+1)}{2}\right)^2 \end{aligned}$$