Sequences and Sums

- Sequences and Progressions
- Summation and its linearity
- Evaluating Sums
- Evaluating Sums - Proofs without words
- Geometric Sums

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Evaluating Sums

Given a summation find a *closed form* formula to evaluate it

A formula involving lower and upper limits only, to output value of the sum

For instance what is the value (in terms of n) of $\sum_{i=1}^{n} i$?

Evaluating Sums

Evaluate $T_n := \sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n - 2 + n - 1 + n$

$$1 + 2 + \overbrace{\overbrace{\overbrace{3 + \cdots\cdots + (n-2)}^{n+1} + (n-1)}^{n+1} + n}^{n+1}$$

- Each pair sums up to $n + 1$
- Number of pairs is $n/2$ ▷ Think about n odd/even

$$T_n = \frac{n(n+1)}{2}$$

Evaluating Sums

$$\text{Evaluate} \quad T_n := \sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n - 2 + n - 1 + n$$

An alternative way to evaluate T_n

$$\begin{array}{rc|ccccccccc} T_n & = & 1 & + & 2 & + & \ldots & + & n-1 & + & n \\ T_n & = & n & + & n-1 & + & \ldots & + & 2 & + & 1 \\ \hline 2T_n & = & n+1 & + & n+1 & + & \ldots & + & n+1 & + & n+1 \end{array}$$

$$2T_n = n(n+1) \quad \Longrightarrow \quad T_n = \frac{n(n+1)}{2}$$

Evaluating Sums

Evaluate $E_n := 2 + 4 + 6 + \ldots + 2(n-2) + 2(n-1) + 2n$

This is

$$E_n := \sum_{i=1}^{n} 2i$$

By linearity of summation

$$E_n = 2\sum_{i=1}^{n} i = 2T_n = 2\frac{n(n+1)}{2} = n(n+1)$$

Square Numbers

We express square numbers in terms of T_n

$$n^2 = T_n + T_{n-1}$$

This can be easily verified as

$$T_n + T_{n-1} = \frac{n(n+1)}{2} + \frac{(n-1)n}{2} = n^2$$

Sum of odd numbers

Evaluate

$$O_n \;:=\; \sum_{i=1}^{n}(2i-1) \;=\; 1+3+5+\ldots+(2n-3)+(2n-1)$$

By linearity of summation

$$O_n = \; 2\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 \;=\; 2T_n - n \;=\; 2\frac{n(n+1)}{2} - n \;=\; n(n+1) - n$$

$$O_n \;=\; 1+3+5+\ldots+2(n-2)-1+2(n-1)-1+2n-1 \;=\; n^2$$

Sum of odd numbers

We got $$O_n := 1 + 3 + \ldots + 2n - 3 + 2n - 1 \;=\; n^2$$

Our earlier expression: $$n^2 \;=\; T_n + T_{n-1} \;=\; \sum_{i=1}^{n} i + \sum_{i=1}^{n-1} i$$

$$O_n \;=\; \overbrace{1 + 2 \ldots + (n-1) + n}^{T_n} \;+\; \overbrace{1 + 2 \ldots + (n-2) + (n-1)}^{T_{n-1}}$$

Another way to look at this is

T_{n-1}			1	+	2	+	3	+	4	+	5	$\ldots$
T_n	1	+	2	+	3	+	4	+	5	+	6	$\ldots$
O_n	1	+	3	+	5	+	7	+	9	+	11	$\ldots$

Sum of squares and cubes

Evaluate sum of squares and sum of cubes of first n positive integers

$$S_n := \sum_{i=1}^{n} i^2 := 1^2 + 2^2 + 3^2 + \ldots + n^2$$

and

$$C_n := \sum_{i=1}^{n} i^3 := 1^3 + 2^3 + 3^3 + \ldots + n^3$$

Sum of Squares

Evaluate

$$S_n := \sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2$$

Sum of Squares

$$(a + 1)^3 = a^3 + 3a^2 + 3a + 1$$

$$1^3 = (0 + 1)^3 = 0^3 + 3 \cdot 0^2 + 3 \cdot 0 + 1$$

Sum of Squares

$$(a+1)^3 = a^3 + 3a^2 + 3a + 1$$

$$\begin{array}{ccccccccccc} 1^3 & = & (0+1)^3 & = & 0^3 & + & 3 \cdot 0^2 & + & 3 \cdot 0 & + & 1 \\ 2^3 & = & (1+1)^3 & = & 1^3 & + & 3 \cdot 1^2 & + & 3 \cdot 1 & + & 1 \end{array}$$

Sum of Squares

$$(a+1)^3 = a^3 + 3a^2 + 3a + 1$$

$$\begin{aligned}
1^3 &= (0+1)^3 = 0^3 + 3\cdot 0^2 + 3\cdot 0 + 1 \\
2^3 &= (1+1)^3 = 1^3 + 3\cdot 1^2 + 3\cdot 1 + 1 \\
3^3 &= (2+1)^3 = 2^3 + 3\cdot 2^2 + 3\cdot 2 + 1
\end{aligned}$$

Sum of Squares

$$(a+1)^3 = a^3 + 3a^2 + 3a + 1$$

$$\begin{array}{lllllllllll}
1^3 & = & (0+1)^3 & = & 0^3 & + & 3\cdot 0^2 & + & 3\cdot 0 & + & 1 \\
2^3 & = & (1+1)^3 & = & 1^3 & + & 3\cdot 1^2 & + & 3\cdot 1 & + & 1 \\
3^3 & = & (2+1)^3 & = & 2^3 & + & 3\cdot 2^2 & + & 3\cdot 2 & + & 1 \\
4^3 & = & (3+1)^3 & = & 3^3 & + & 3\cdot 3^2 & + & 3\cdot 3 & + & 1
\end{array}$$

Sum of Squares

$$(a+1)^3 = a^3 + 3a^2 + 3a + 1$$

$$\begin{array}{ccccccccccc}
1^3 & = & (0+1)^3 & = & 0^3 & + & 3\cdot 0^2 & + & 3\cdot 0 & + & 1 \\
2^3 & = & (1+1)^3 & = & 1^3 & + & 3\cdot 1^2 & + & 3\cdot 1 & + & 1 \\
3^3 & = & (2+1)^3 & = & 2^3 & + & 3\cdot 2^2 & + & 3\cdot 2 & + & 1 \\
4^3 & = & (3+1)^3 & = & 3^3 & + & 3\cdot 3^2 & + & 3\cdot 3 & + & 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
(n+1)^3 & = & (n+1)^3 & = & n^3 & + & 3\cdot n^2 & + & 3\cdot n & + & 1 \\
\hline
\end{array}$$

Sum of Squares

$$(a+1)^3 = a^3 + 3a^2 + 3a + 1$$

$$\begin{array}{ccccccccccc}
1^3 & = & (0+1)^3 & = & 0^3 & + & 3\cdot 0^2 & + & 3\cdot 0 & + & 1 \\
2^3 & = & (1+1)^3 & = & 1^3 & + & 3\cdot 1^2 & + & 3\cdot 1 & + & 1 \\
3^3 & = & (2+1)^3 & = & 2^3 & + & 3\cdot 2^2 & + & 3\cdot 2 & + & 1 \\
4^3 & = & (3+1)^3 & = & 3^3 & + & 3\cdot 3^2 & + & 3\cdot 3 & + & 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
(n+1)^3 & = & (n+1)^3 & = & n^3 & + & 3\cdot n^2 & + & 3\cdot n & + & 1 \\
\hline
\sum & & & & & & & & & &
\end{array}$$

Sum of Squares

$$(a+1)^3 = a^3 + 3a^2 + 3a + 1$$

$$\begin{array}{ccccccccccc}
1^3 & = & (0+1)^3 & = & 0^3 & + & 3\cdot 0^2 & + & 3\cdot 0 & + & 1 \\
2^3 & = & (1+1)^3 & = & 1^3 & + & 3\cdot 1^2 & + & 3\cdot 1 & + & 1 \\
3^3 & = & (2+1)^3 & = & 2^3 & + & 3\cdot 2^2 & + & 3\cdot 2 & + & 1 \\
4^3 & = & (3+1)^3 & = & 3^3 & + & 3\cdot 3^2 & + & 3\cdot 3 & + & 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
(n+1)^3 & = & (n+1)^3 & = & n^3 & + & 3\cdot n^2 & + & 3\cdot n & + & 1 \\
\hline
\sum & & \sum_{i=0}^{n}(i+1)^3 & = & & & & & & &
\end{array}$$

Sum of Squares

$$(a+1)^3 = a^3 + 3a^2 + 3a + 1$$

$$\begin{array}{ccccccccccc}
1^3 & = & (0+1)^3 & = & 0^3 & + & 3\cdot 0^2 & + & 3\cdot 0 & + & 1 \\
2^3 & = & (1+1)^3 & = & 1^3 & + & 3\cdot 1^2 & + & 3\cdot 1 & + & 1 \\
3^3 & = & (2+1)^3 & = & 2^3 & + & 3\cdot 2^2 & + & 3\cdot 2 & + & 1 \\
4^3 & = & (3+1)^3 & = & 3^3 & + & 3\cdot 3^2 & + & 3\cdot 3 & + & 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
(n+1)^3 & = & (n+1)^3 & = & n^3 & + & 3\cdot n^2 & + & 3\cdot n & + & 1 \\
\hline
\sum & & \sum_{i=0}^{n}(i+1)^3 & = & \sum_{i=0}^{n} i^3 & + & 3\sum_{i=0}^{n} i^2 & + & 3\sum_{i=0}^{n} i & + & n+1
\end{array}$$

Sum of Squares

$$(a+1)^3 = a^3 + 3a^2 + 3a + 1$$

$$\begin{array}{ccccccccccccc}
1^3 & = & (0+1)^3 & = & 0^3 & + & 3\cdot 0^2 & + & 3\cdot 0 & + & 1 \\
2^3 & = & (1+1)^3 & = & 1^3 & + & 3\cdot 1^2 & + & 3\cdot 1 & + & 1 \\
3^3 & = & (2+1)^3 & = & 2^3 & + & 3\cdot 2^2 & + & 3\cdot 2 & + & 1 \\
4^3 & = & (3+1)^3 & = & 3^3 & + & 3\cdot 3^2 & + & 3\cdot 3 & + & 1 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
(n+1)^3 & = & (n+1)^3 & = & n^3 & + & 3\cdot n^2 & + & 3\cdot n & + & 1 \\
\hline
\sum & & \sum_{i=0}^{n}(i+1)^3 & = & \sum_{i=0}^{n} i^3 & + & 3\sum_{i=0}^{n} i^2 & + & 3\sum_{i=0}^{n} i & + & n+1
\end{array}$$

Sum of Squares

$$\sum_{i=0}^{n}(i+1)^3 \qquad = \sum_{i=0}^{n} i^3 + 3\sum_{i=0}^{n} i^2 + 3\sum_{i=0}^{n} i + n+1$$

Sum of Squares

$$\sum_{i=0}^{n}(i+1)^3 = \sum_{i=0}^{n} i^3 + 3\sum_{i=0}^{n} i^2 + 3\sum_{i=0}^{n} i + n+1$$

$$\sum_{i=1}^{n+1} i^3 = \sum_{i=1}^{n} i^3 + 3\sum_{i=0}^{n} i^2 + 3\sum_{i=0}^{n} i + n+1$$

Sum of Squares

$$\sum_{i=0}^{n}(i+1)^3 = \sum_{i=0}^{n} i^3 + 3\sum_{i=0}^{n} i^2 + 3\sum_{i=0}^{n} i + n+1$$

$$\sum_{i=1}^{n+1} i^3 = \sum_{i=1}^{n} i^3 + 3\sum_{i=0}^{n} i^2 + 3\sum_{i=0}^{n} i + n+1$$

$$\sum_{i=1}^{n} i^3 + (n+1)^3 = \sum_{i=1}^{n} i^3 + 3\sum_{i=0}^{n} i^2 + 3\sum_{i=0}^{n} i + n+1$$

Sum of Squares

$$\sum_{i=0}^{n}(i+1)^3 = \sum_{i=0}^{n} i^3 + 3\sum_{i=0}^{n} i^2 + 3\sum_{i=0}^{n} i + n+1$$

$$\sum_{i=1}^{n+1} i^3 = \sum_{i=1}^{n} i^3 + 3\sum_{i=0}^{n} i^2 + 3\sum_{i=0}^{n} i + n+1$$

$$\cancel{\sum_{i=1}^{n} i^3} + (n+1)^3 = \cancel{\sum_{i=1}^{n} i^3} + 3\underbrace{\sum_{i=0}^{n} i^2}_{S_n} + 3\underbrace{\sum_{i=0}^{n} i}_{T_n} + n+1$$

$$\implies 3S_n = 3\sum_{i=1}^{n} i^2 = (n+1)^3 - 3\sum_{i=0}^{n} i - (n+1)$$

$$\implies S_n = \sum_{i=1}^{n} i^2 = (n+1)\left(n+1/2\right)n = \frac{n(n+1)(2n+1)}{6}$$

Sum of Cubes

Evaluate $$C_n := \sum_{i=1}^{n} i^3 = 1^3 + 2^3 + 3^3 + \ldots + n^3$$

$$1^3 + 2^3 + \ldots (n-1)^3 + n^3 = (T_n)^2$$

$$= \left(\frac{n(n+1)}{2}\right)^2$$

Important Sums

$$\sum_{i=1}^{n} i = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} 2i = 2 + 4 + \ldots + 2n = n(n+1)$$

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\sum_{i=1}^{n} 2i - 1 = 1 + 3 + \ldots + 2n - 1 = n^2$$

Evaluating Sums

Evaluate the following sum in terms of appropriate variables

$$\sum_{i=3}^{n} 2(i-1)^2$$

Let $j = i - 1$, we get

$$\sum_{j=2}^{n-1} 2j^2 \ = \ 2\sum_{j=2}^{n-1} j^2 \ = \ 2\Bigg(\sum_{j=1}^{n-1} j^2 - 1^2\Bigg) \ = \ 2\Bigg(\frac{(n-1)(n)(2n-1)}{6} - 1^2\Bigg)$$

Evaluating Sums

Evaluate the following sum in terms of appropriate variables

$$\sum_{n=k}^{1} 3n^3 = 3\sum_{n=k}^{1} n^3 = 3\sum_{n=1}^{k} n^3 = 3\left(\frac{k(k+1)}{2}\right)^2$$

Evaluating Sums

Evaluate the following sum in terms of appropriate variables

ICP 7-9 $$\sum_{k=1}^{n}(k^2+2k+1)$$

ICP 7-10 $$\sum_{i=1}^{m}3i^3+4i^2$$

Telescoping Sum

Let $\{a_i\}$ be a sequence. Evaluate

$$\sum_{i=0}^{n} a_{i+1} - a_i$$

$$\begin{aligned}
&\sum_{i=0}^{n} a_{i+1} - a_i \\
&= (a_1 - a_0) + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_n - a_{n-1}) + (a_{n+1} - a_n) \\
&= (\cancel{a_1} - a_0) + (a_2 - \cancel{a_1}) + (a_3 - a_2) + \cdots + (a_n - a_{n-1}) + (a_{n+1} - a_n) \\
&= (\cancel{a_1} - a_0) + (\cancel{a_2} - \cancel{a_1}) + (\cancel{a_3} - \cancel{a_2}) + \cdots + (\cancel{a_n} - \cancel{a_{n-1}}) + (a_{n+1} - \cancel{a_n}) \\
&= a_{n+1} - a_0
\end{aligned}$$

Telescoping Sum

Evaluate $$\sum_{k=1}^{n} \frac{1}{k(k+1)}$$

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1}$$

$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \left(\frac{1}{1} - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cdots + \left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$= \frac{1}{1} - \frac{1}{n+1} = \frac{n}{n+1}$$