Sequences and Sums

- Sequences and Progressions
- Summation and its linearity
- Evaluating Sums
- Evaluating Sums - Proofs without words
- Geometric Sums

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Sequences, Strings and Progressions

- A sequence is an ordered list —– could be finite or infinite
- An infinite sequence is a function $\mathbb{N} \mapsto S$
- A finite sequence of length n is a function $\{1, 2, \ldots, n\} \mapsto S$
- f represents the order of elements in S
- Finite sequences over a fixed alphabet are called strings
- Geometric progression is a sequence of numbers, where the next term is obtained by multiplying the previous term with the common ratio r
- Arithmetic progression is a sequence of numbers, where the next term is obtained by adding the previous term with the common difference d

Summation

The symbol $\sum$ converts a sequence of numbers into a **sum**

$$\sum_{i=0}^{n} a_i = a_0 + a_1 + a_2 + \cdots + a_n$$

pronounced as "sum of a_i, where i goes from 0 to n"

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
a_0	a_1	a_2	a_3												a_{15}	a_{16}	a_{17}

$$\sum_{i=0}^{17} a_i$$

Summation

The symbol $\sum$ converts a sequence of numbers into a **sum**

$$\sum_{i=\ell}^{m} a_i$$

pronounced as "sum of a_i, where i goes from ℓ to m"

- i: index of summation
- ℓ, m: lower & upper limit

0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
a_0	a_1	a_2	a_3							a_{10}					a_{15}	a_{16}	a_{17}

$$\sum_{i=2}^{10} a_i$$

Summation

$0 + 1 + 2 + 3 + 4 + \ldots + n$

$$\sum_{i=0}^{n} i$$

```
sum = 0;
for(i=0;i<=n;i++)
{
    sum = sum + i;
}
```

First n terms of

$0 + 3 + 8 + 15 + 24 + \ldots$

$$\sum_{i=1}^{n} i^2 - 1$$

```
sum = 0;
for(i=1;i<=n;i++)
{
    sum = sum + i * i - 1;
}
```

Summation

Write the following in expanded form

$$\sum_{i=3}^{6} 2i^2 = 2(3^2) + 2(4^2) + 2(5^2) + 2(6^2)$$

$$\sum_{n=4}^{1} n^3 = 4^3 + 3^3 + 2^3 + 1^3$$

ICP 7-7

$$\sum_{i=1}^{3}(i^2 + 2i + 1) = \sum_{i=1}^{3}(i+1)^2 = 2^2 + 3^2 + 4^2$$

ICP 7-8

$$2\sum_{j=6}^{3} j^2 = 2(6^2) + 2(5^2) + 2(4^2) + 2(3^2)$$

Summation: Change of variable

Changing the variables sometimes helps evaluating the sum by bringing it into a familiar form

$$\sum_{j=1}^{6}(j+3)^3 \quad = \quad 4^3+5^3+6^3+7^3+8^3+9^3$$

Let $k = j + 3$ then

$$\sum_{j=1}^{6}(j+3)^3 = \sum_{k=4}^{9} k^3$$

Be careful in evaluating the lower and upper limits

Linearity of Summation

A function $f(\cdot)$ is linear if for all real numbers a and b

$$f(ax + by) \; = \; a \cdot f(x) + b \cdot f(y)$$

Summation is linear!

$$\sum_{i=1}^{n} (p \cdot a_i + q \cdot b_i) \;\; = \;\; p \cdot \sum_{i=1}^{n} a_i + q \cdot \sum_{i=1}^{n} b_i$$

In the simpler form $(q = 0)$, we get

$$\sum_{i=1}^{n} p \cdot a_i \;\; = \;\; p \cdot \sum_{i=1}^{n} a_i$$

Linearity of Summation

$$\sum_{i=1}^{3} 7i^2 \;=\; 7(1)^2 + 7(2)^2 + 7(3)^2 \;=\; 7(1^2 + 2^2 + 3^2) \;=\; 7\sum_{i=1}^{3} i^2$$

$$\begin{aligned}
\sum_{i=1}^{n} p \cdot a_i \;&=\; p \cdot a_1 + p \cdot a_2 + \ldots + p \cdot a_n \\
&=\; \underbrace{a_1 + \ldots + a_1}_{p \text{ times}} + \underbrace{a_2 + \ldots + a_2}_{p \text{ times}} + \;\ldots\; + \underbrace{a_n + \ldots + a_n}_{p \text{ times}} \\
&=\; \underbrace{(a_1 + \ldots + a_n) \;+\; (a_1 + \ldots + a_n) \;+\; \ldots + (a_1 + \ldots + a_n)}_{p \text{ times}} \\
&=\; \underbrace{\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} a_i + \;\ldots\; + \sum_{i=1}^{n} a_i}_{p \text{ times}} \;=\; p \cdot \sum_{i=1}^{n} a_i
\end{aligned}$$

Sum Applications

When **Friedrich Gauss** was nine years old, their teacher wanted to pass time so asked the class to evaluate

$$1 + 2 + 3 + \ldots + 999 + 1000$$

Gauss solved the problem in 30 seconds

It seems a lot of stuff is attributed to Gauss – either he was really smart or he had a great press agent.

Sidenote in 'Concrete Mathematics'

Sum Applications

Algorithm Sort an array A of n numbers

for $i = 1$ to n **do**
  for $j = i + 1$ to n **do**
    if $A[i] > A[j]$ **then**
      EXCHANGE$(A[i], A[j])$

i	j's compared with	number of comparisons
1	$2, 3, \ldots, n$	$n-1$
2	$3, 4, \ldots, n$	$n-2$
$\vdots$	$\vdots$	$\vdots$
$n-2$	$n-1, n$	2
$n-1$	n	1

Number of comparisons $= 1 + 2 + \ldots + n - 1$

Sum Applications

Organizing Tournaments

- n teams are participating in Round 1 of a soccer tournament
- Every team play every other team exactly once
- Each game is refereed by a professional
- The referee charges \$1 per game

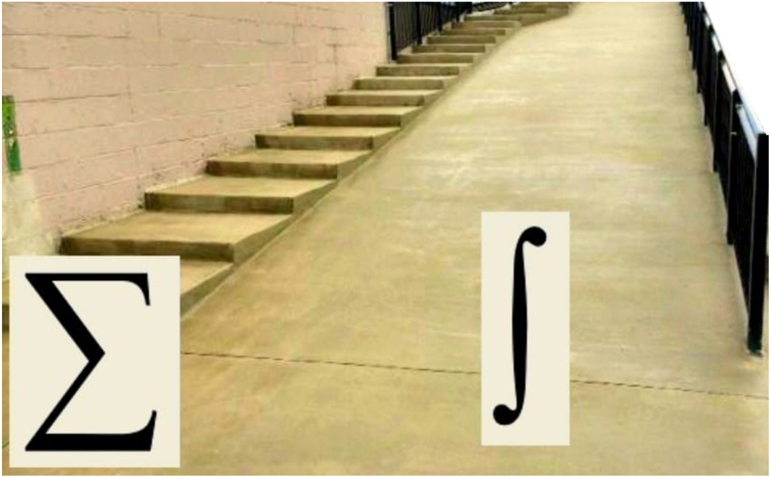
What is the total payment to the referee?

Organizing Tournaments

Team	Teams played with	num games	num games
1	$2, 3, 4, \ldots, n-1, n$	$n-1$	$n-1$
2	$1, 3, 4, \ldots, n-1, n$	$n-1$	$n-2$
3	$1, 2, 4, \ldots, n-1, n$	$n-1$	$n-3$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	$1, 2, \ldots n-2, n$	$n-1$	1
n	$1, 2, \ldots n-2, n-1$	$n-1$	0

num games $= 1 + 2 + \ldots + n - 1$

Summation is the discrete analog of Integration



Product

Occasionally, we use the product.

The symbol $\prod$ converts a sequence of numbers into a **product**

$$\prod_{i=0}^{n} a_i = a_0 \times a_1 \times a_2 \times \cdots \times a_n$$

pronounced as "product of a_i, where i goes from 0 to n"

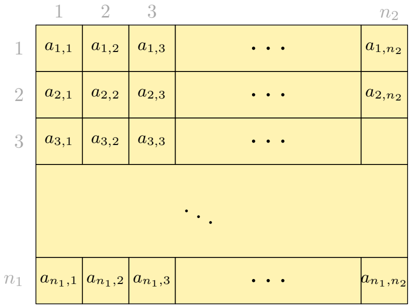
```
answer = 1;
for(i=0;i<=n;i++)
{
    answer = answer * a[i];
}
```

Double or Nested Summation

For a sequence of numbers $a_{i,j}$ (a sequence indexed by two variables) or 2D array or matrix, double summation converts it into a **sum**

$$\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} a_{i,j}$$

$$\begin{aligned}
&= a_{1,1} + a_{1,2} + \cdots + a_{1,n_2} \\
&+ a_{2,1} + a_{2,2} + \cdots + a_{2,n_2} \\
&+ a_{3,1} + a_{3,2} + \cdots + a_{3,n_2} \\
&\vdots \\
&\vdots \\
&+ a_{n_1,1} + a_{n_1,2} + \cdots + a_{n_1,n_2}
\end{aligned}$$



Double or Nested Summation

For a sequence of numbers $a_{i,j}$ (a sequence indexed by two variables) or 2D array or matrix, double summation converts it into a **sum**

$$\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} a_{i,j}$$

$$= a_{1,1} + a_{1,2} + \cdots + a_{1,n_2}$$
$$+ a_{2,1} + a_{2,2} + \cdots + a_{2,n_2}$$
$$+ a_{3,1} + a_{3,2} + \cdots + a_{3,n_2}$$
$$\vdots$$
$$\vdots$$
$$+ a_{n_1,1} + a_{n_1,2} + \cdots + a_{n_1,n_2}$$

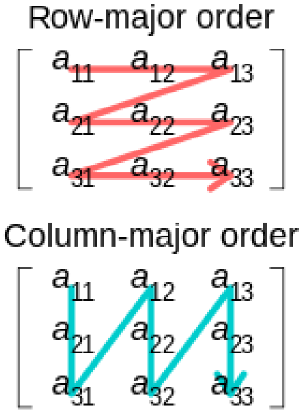
	1	2	3		n_2
1	$a_{1,1}$	$a_{1,2}$	$a_{1,3}$	$\cdots$	a_{1,n_2}
2	$a_{2,1}$	$a_{2,2}$	$a_{2,3}$	$\cdots$	a_{2,n_2}
3	$a_{3,1}$	$a_{3,2}$	$a_{3,3}$	$\cdots$	
			$\ddots$		
n_1	$a_{n_1,1}$	$a_{n_1,2}$	$a_{n_1,3}$	$\cdots$	a_{n_1,n_2}

```
sum = 0;
for(i=1;i<=n1;i++)
{
    for(j=1;j<=n2;j++)
    {
        sum = sum + a[i][j];
    }
}
```

Double or Nested Summation

For a sequence of numbers $a_{i,j}$ (a sequence indexed by two variables) or 2D array or matrix, double summation converts it into a **sum**

$$\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} a_{i,j} = \sum_{j=1}^{n_2}\sum_{i=1}^{n_1} a_{i,j}$$

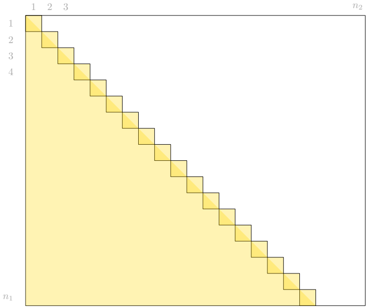


Double or Nested Summation

For a sequence of numbers $a_{i,j}$ (a sequence indexed by two variables) or 2D array or matrix, double summation converts it into a **sum**

$$\sum_{i=1}^{n_1} \sum_{j=1}^{i} a_{i,j}$$

$$= a_{1,1}$$
$$+ a_{2,1} + a_{2,2}$$
$$+ a_{3,1} + a_{3,2} + a_{3,3}$$
$$+ a_{4,1} + a_{4,2} + a_{4,3} + a_{4,4}$$
$$\vdots$$
$$+ a_{n_1,1} + a_{n_1,2} + \cdots + a_{n_1,n_1}$$



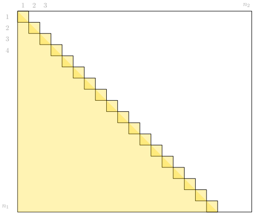
Assuming $n_1 \leq n_2$

Double or Nested Summation

For a sequence of numbers $a_{i,j}$ (a sequence indexed by two variables) or 2D array or matrix, double summation converts it into a **sum**

$$\sum_{i=1}^{n_1} \sum_{j=1}^{i} a_{i,j}$$

$$\begin{aligned} &= a_{1,1} \\ &+ a_{2,1} + a_{2,2} \\ &+ a_{3,1} + a_{3,2} + a_{3,3} \\ &+ a_{4,1} + a_{4,2} + a_{4,3} + a_{4,4} \\ &\quad\vdots \\ &+ a_{n_1,1} + a_{n_1,2} + \cdots + a_{n_1,n_1} \end{aligned}$$



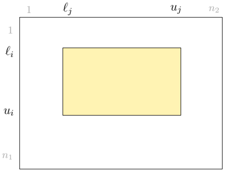

```
sum = 0;
for(i=1;i<=n1;i++)
{
    for(j=1;j<=i;j++)
    {
        sum = sum + a[i][j];
    }
}
```

Assuming $n_1 \leq n_2$

Double or Nested Summation

For a sequence of numbers $a_{i,j}$ (a sequence indexed by two variables) or 2D array or matrix, double summation converts it into a **sum**

$$\sum_{i=\ell_i}^{u_i} \sum_{j=\ell_j}^{u_j} a_{i,j}$$



Assuming

$1 \leq \ell_i \leq u_i \leq n_1$

and

$1 \leq \ell_j \leq u_j \leq n_2$

```
sum = 0;
for(i=l_i;i<=u_i;i++)
{
    for(j=l_j;j<=u_j;j++)
    {
        sum = sum + a[i][j];
    }
}
```