Sequences and Sums

- Sequences and Progressions
- Summation and its linearity
- Evaluating Sums
- Evaluating Sums - Proofs without words
- Geometric Sums

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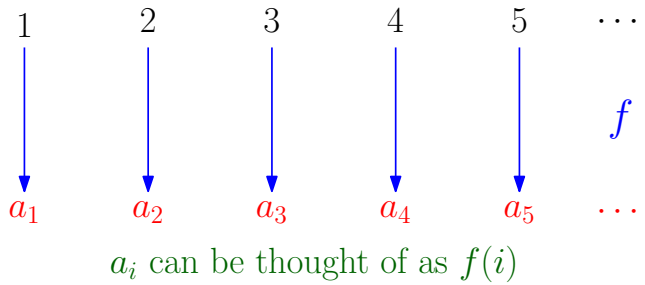
Sequences

A sequence is an ordered list —— could be finite or infinite

- $\{a_i\}$ denotes the sequence
- a_i is the i^{th} term of the sequence ▷ general term



Given a (multi)set S $(n = |S|)$, a sequence is a function $f : \{1, 2, \ldots, n\} \mapsto S$

- An infinite sequence is a function $\mathbb{N} \mapsto S$
- f represents the order of elements in S ▷ assigns indices to S
- Be careful whether you start indexing from $i = 0$ or $i = 1$

Sequences with Patterns

What is the general term of the following sequences?

$1, 2, 3, 4, 5, 6, \ldots$

▷ $a_i = i$

$2, 4, 6, 8, 10, \ldots$

▷ $a_i = 2i$

$1, 3, 5, 7, 9, \ldots$

▷ $a_i = 2i - 1$

ICP 7-1 $2, 5, 10, 17, 26, \ldots$

▷ $a_i = i^2 + 1$

ICP 7-2 $2, 8, 26, 80, 242, \ldots$

▷ $a_i = 3^i - 1$

Sequences with Patterns

List the first 5 terms of each of these sequences.

The sequence starting with 10 and each term is obtained by subtracting 3 from the previous term

▷ $10, 7, 4, 1, -2$

The sequence whose nth term is the sum of the first n positive integers

▷ $1, 3, 6, 10, 15$

The sequence whose nth term is $3^n - 2^n$

▷ $1, 5, 19, 65, 211$

ICP 7-3 The sequence whose nth term is the largest integer whose binary expansion has n bits (write your answer in decimal notation)

▷ $1, 3, 7, 15, 31$

ICP 7-4 The sequence whose first two terms are 1 and 5 and each succeeding term is the sum of the two previous terms

▷ $1, 5, 6, 11, 17$

Strings

- Finite sequences are called strings
- Length of a string is the number of terms it consists of
- The empty string contains no term

  ▷ Length of the empty string is 0
- Alphabet is the set of all possible terms

$$\text{Alphabet} = \{0, 1\} \text{ —— bit strings}$$

Geometric Progressions

A geometric progression is a sequence of the form

$$a, ar, ar^2, \ldots, ar^i, ar^{i+1} \ldots$$

where a and r are real numbers

$$\frac{ar^{i+1}}{ar^i} = r$$

- the ratio of consecutive terms is called the common ratio
- a is called the initial term
- The next term is obtained by multiplying the previous term with r

Arithmetic Progressions

An arithmetic progression is a sequence of the form

$$a, a+d, a+2d, \ldots, a+id, a+(i+1)d, \ldots$$

where a and d are real numbers

$$(a+(i+1)d) - (a+id) \;=\; d$$

- the difference of consecutive terms is called the common difference
- a is called the initial term
- The next term is obtained by adding the previous term with d

Progression Examples

$$1, -1, 1, -1, 1, -1, 1, -1, \ldots$$

GP, $a_1 = 1$, common ratio $= -1$, $a_i = (-1)^{i-1}$

$$3, 9, 27, 81, 243, \ldots$$

GP, $a_1 = 3$, common ratio $= 3$, $a_i = 3^i$

$$1, 1/2, 1/4, 1/8, 1/16, \ldots$$

GP, $a_1 = 1$, common ratio $= 1/2$, $a_i = (1/2)^{i-1}$

Progression Examples

$$1, 4, 7, 10, 13, 16, \ldots$$

AP, $a_1 = 1$, common difference $= 3$, $a_i = 1 + 3(i-1)$

$$6, 2, -2, -6, \ldots$$

AP, $a_1 = 6$, common difference $= -4$, $a_i = 6 - 4(i-1)$

Progression Examples

For the following sequences, write the type of progression, initial term, common ratio/difference and general term?

ICP 7-5 $1\,,\ -2\,,\ 4\,,\ -8\,,\ 16\,,\ -32\,,\ \ldots$

ICP 7-6 $-2.7\,,\ 0.1\,,\ 2.9\,,\ 5.8\,,\ \ldots$

Some Common Sequences

a_i	**First** 10 **Terms**
i^2	$1, 4, 9, 16, 25, 36, 49, 64, 81, 100, \ldots$
i^3	$1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, \ldots$
i^4	$1, 16, 81, 256, 1296, 2401, 4096, 6561, 10000, \ldots$
2^i	$2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, \ldots$
3^i	$3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049$
$i!$	$1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800$

Sequences, Strings and Progressions

- A sequence is an ordered list —– could be finite or infinite
- An infinite sequence is a function $\mathbb{N} \to S$
- A finite sequence of length is a function $\{1, 2, \ldots, n\} \to S$
- f represents the order of elements in S
- Finite sequences over a fixed alphabet are called strings
- Geometric progression is a sequence of numbers, where the next term is obtained by multiplying the previous term with the common ratio r
- Arithmetic progression is a sequence of numbers, where the next term is obtained by adding the previous term with the common difference d