Relations

- Relations: Definition and Notation
- Properties of Relations
- Combining Relations
- Operations on Relations: Projection and Join
- Equivalence Relations and Equivalence Classes
- Partial Order

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Relations on a Set

A **(binary) relation** from X to Y is a subset of $X \times Y$

A **(binary) relation on a set** X is subset of $X \times X$ (relation from X to X)

A relation R on a set X is **reflexive** if $(a, a) \in R$ for every element $a \in X$

A relation R on a set X is **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in X$

A relation R on a set X is **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$

Equivalence Relation

Equivalence Relation

A relation R on a set X is an equivalence relation if it is

1. reflexive
2. symmetric, and
3. transitive

- Relates "similar" elements
- Generalize "equality"

Equivalence Relation

Consider the relation '$=$' on the set of integers

$$R = \{(a,b) \mid a = b\}$$

- $a = a$ ▷ reflexive
- $a = b \leftrightarrow b = a$ ▷ symmetric
- $(a = b \land b = c) \rightarrow (a = c)$ ▷ transitive

Hence '$=$' is an equivalence relation

For any reasonable notion of equivalence these three properties are needed

Equivalence Relation

$$R = \{(a, b) \mid a = b \text{ or } a = -b\}$$

Is R an equivalence relation on the set of integers?

ICP 6-26 Reflexive?

ICP 6-27 Symmetric?

ICP 6-28 Transitive?

Equivalence Relation

Let S be a set of strings

Let R_n is a relation on S

$(s, t) \in R_n$ if $s = t$ OR

$len(s) \geq n$ and $len(t) \geq n$ AND the first n characters of s and t are same

- $(abc, abd) \notin R_4$
- $(xyz, xyz) \in R_4$
- $(xyz1, xyz1) \in R_4$
- $(string, string) \in R_4$
- $(string, strike) \in R_4$
- $(abc, abd) \notin R_5$
- $(compiler, compilation) \in R_5$
- $(compiler, compilation) \in R_6$
- $(string, string) \in R_6$
- $(string, strike) \notin R_6$

Verify that R_n is an equivalence relation

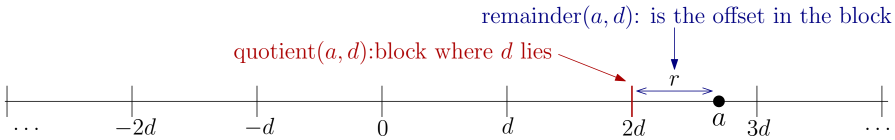
Identifiers in the C programming language R_{31}

Proof using Principle of Well Ordering

Theorem (The Division Algorithm)

Let a be an integer and d a positive integer. Then there are unique integers q and r, with $0 \leq r < d$ such that $a = dq + r$

- q is called the quotient
- r is called the remainder
- d is called the divisor
- a is called the dividend



The Division Algorithm

Theorem (The Division Algorithm)

Let a be an integer and d a positive integer. Then there are unique integers q and r, with $0 \leq r < d$ such that $a = dq + r$

quotient and remainder when 11 is divided by 5

$$11 = 5 \times 2 + 1$$

Notation:

$q = 2 = 11 \text{ div } 5$
$r = 1 = 11 \text{ mod } 5$

The Division Algorithm

Theorem (The Division Algorithm)

Let a be an integer and d a positive integer. Then there are unique integers q and r, with $0 \leq r < d$ such that $a = dq + r$

quotient and remainder when -11 is divided by 5

$$-11 \;=\; 5 \times -3 + 4$$

Notation:

$q = -3 \;=\; -11 \;\; \text{div}\; 5$
$r = 4 \;=\; -11 \;\; \text{mod}\; 5$

The Division Algorithm

Theorem (The Division Algorithm)

Let a be an integer and d a positive integer. Then there are unique integers q and r, with $0 \leq r < d$ such that $a = dq + r$

$11 \;=\; 5 \times 2 + 1$

Given positive a, repeatedly subtract d until what remains r is $0 \leq r < d$

$-11 \;=\; 5 \times -3 + 4$

Given negative a, repeatedly add d until what remains r is $0 \leq r < d$

The Remainder Operator

$$a \text{ mod } d = a \,\%\, d$$

The remainder when a is divided by d

$$\begin{array}{lll} \text{If} & a = qd + r & (0 \leq r < d), \\ \text{then} & r = a \text{ mod } d & \\ \text{and} & q = a \text{ div } d & \end{array}$$

Equivalence Relation

Definition (Congruence (mod k))

$x \equiv y \pmod{k}$ if and only if $x \bmod k = y \bmod k$

$$R = \{(a,b) : a \equiv b \pmod{3}\}$$

ICP 6-29 Is R an equivalence relation on integers?

1. Reflexive: obvious
2. Symmetric: Suppose $a = 3q_a + r_a$ and $b = 3q_b + r_b$

   If $(a,b) \in R$, then $r_a = r_b$. So $r_b = r_a$ and $(b,a) \in R$
3. Transitive: Suppose $a = 3q_a + r_a$ and $b = 3q_b + r_b$, $c = 3q_c + r_c$

   If $(a,b),(b,c) \in R$, then $r_a = r_b, r_b = r_c$. So $r_a = r_c$ and $(a,c) \in R$

Equivalence Relation

$$R = \{(a,b) : a \mid b\}$$

ICP 6-30 Is R an equivalence relation on integers?

1. Reflexive: a divides a

2. Transitive: If $(a,b),(b,c) \in R$, then $b = a * q_1$ and $c = b * q_2$

   So $c = a * q_1 q_2 \implies a \mid c$ $\triangleright$ hence $(a,c) \in R$

3. Symmetric: $2|4$ but $4 \nmid 2$

Equivalence Relation

ICP 6-31

Suppose that R is the relation on the set of strings of English letters such that $(a, b) \in R$ if and only if $len(a) = len(b)$, where $len(x)$ is the length of the string x.

Is R an equivalence relation?

Equivalence Classes

Let R be an equivalence relation on a set X

For $a \in X$, the set of all elements related to a is called the equivalence class of a

- $[a]_R$ denotes the equivalence class of a
- When R is clear, it is denoted by $[a]$
- If $b \in [a]$, then b is called a representative of this equivalence class

Equivalence Classes

$$R = \{(a, b) \mid a = b \text{ or } a = -b\}$$

Equivalence class of 7?

$[7] = \{7, -7\}$

Equivalence class of -7?

$[-7] = \{7, -7\}$

Equivalence class of -3?

$[-3] = \{3, -3\}$

Equivalence class of 0?

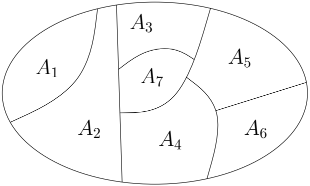
$[0] = \{0\}$

Equivalence Classes and Partitions

A partition of a set S is a collection of disjoint non-empty subsets of S, with S as their union

A collection of subsets A_i, $i \in I$ (where I is an index set) forms a partition of S if and only if

1. $A_i \neq \emptyset$ for $i \in I$
2. $A_i \cap A_j = \emptyset$ if $i \neq j$
3. $\bigcup_{i \in I} A_i = S$



Equivalence Classes and Partitions

Theorem

Let R be an equivalence relation on a set S. Then the equivalence classes of R form a partition of S.

Conversely, given a partition $\{A_i : i \in I\}$ of the set S, there is an equivalence relation R that has the sets $A_i, i \in I$, as its equivalence classes

Equivalence Classes and Partitions

$$R = \{(a,b) : a \equiv b \pmod{3}\}$$

R is an equivalence relation on integers

R is called congruence relation modulo 3.

Equivalence class $\sim$ congruence class modulo 3

3 possible remainders $\implies$ 3 congruence classes

$$[0] = \{\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots\}$$
$$[1] = \{\ldots, -8, -5, -2, 1, 4, 7, 10 \ldots\}$$
$$[2] = \{\ldots, -7, -4, -1, 2, 5, 8, 11, \ldots\}$$

Equivalence Classes and Partitions

ICP 6-32

What are equivalence classes of 0, 1, 2, and 3 of congruence modulo 4?