Relations

- Relations: Definition and Notation
- Properties of Relations
- Combining Relations
- Operations on Relations: Projection and Join
- Equivalence Relations and Equivalence Classes
- Partial Order

Imdad ullah Khan

Combining Relations

- Relations from A to B are **subsets** of $A \times B$
- All set operations can be performed on them

$A = \{1, 2, 3\}$ and $B = \{1, 2\}$

Relation $R := \{(1, 2), (3, 2)\}$

Relation $S := \{(1, 1), (2, 2), (1, 2), (3, 2)\}$

- $R \cup S = \{(1, 2), (3, 2), (1, 1), (2, 2)\}$
- $R \cap S = \{(1, 2), (3, 2)\}$
- $R \setminus S = \{\} = \emptyset$
- $S \setminus R = \{(1, 1), (2, 2)\}$

Combining Relations

Students := {All LUMS CS Students}

Courses := {All LUMS CS Courses}

$R := \{(s, c) :$ student s passed course $c\}$

$S := \{(s, c) :$ student s needs course c to graduate$\}$

ICP 6-12 Briefly Describe (in colloquial terms if possible)

- $R \cup S$
- $R \cap S$
- $R \setminus S$
- $S \setminus R$

Composing Relations

Like functions, binary relations can be composed

- R is a relation from A to B
- S is a relation from B to C
- The composite of R and S is a relation from A to C consisting of $(a, c) \in A \times C$, if there exists $b \in B$ such that $(a, b) \in R$ and $(b, c) \in R$

Denoted by $R \circ S$

If $(a, b) \in R$ and $(b, c) \in S$, then $(a, c) \in R \circ S$

Composing Relations

If $(a, b) \in R$ and $(b, c) \in S$, then $(a, c) \in R \circ S$

$A = \{1, 2, 3, 4\}$

$R = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$

$S = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$

$$R \circ S = \{(2, 4), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)\}$$

R relates an element x to $(5 - x)$

S relates $(5 - x)$ to elements larger than $(5 - x)$

$$R \circ S = \{(a, b) : b > 5 - a\}$$

Composing Relations

Let $A = \{1, 2, 3, 4\}$. Consider the following three relations on A

- $R_1 = \{(1,1),(1,2),(2,2),(2,1),(3,3)\}$
- $R_2 = \{(1,1)\}$
- $R_3 = \{(1,3),(3,2),(2,1)\}$

Write the following relations

$$R_1 \circ R_2 = \{(1,1),(2,1)\}$$

$$R_1 \circ R_3 = \{(1,3),(1,1),(2,1),(2,3),(3,2)\}$$

ICP 6-13 $R_2 \circ R_1$

ICP 6-14 $R_2 \circ R_3$

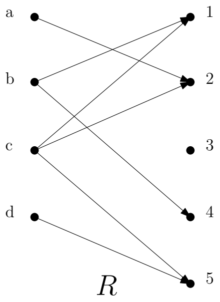
ICP 6-15 $R_3 \circ R_1$

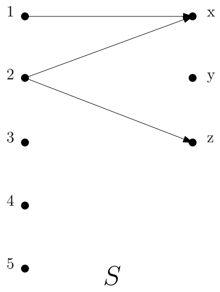
ICP 6-16 $R_3 \circ R_2$

Composing Relations

If $(a, b) \in R$ and $(b, c) \in S$, then $(a, c) \in R \circ S$

- R is a relation from A to B, S from B to C, and $R \circ S$ from A to C
- Draw R and S with the common domain (B)
- Drawing all possible shortcuts

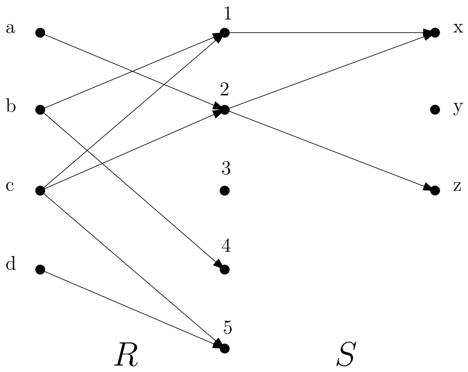




Composing Relations

If $(a, b) \in R$ and $(b, c) \in S$, then $(a, c) \in R \circ S$

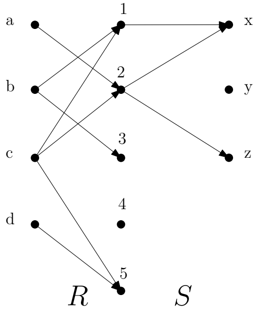
- R is a relation from A to B, S from B to C, and $R \circ S$ from A to C
- Draw R and S with the common domain (B)
- Drawing all possible shortcuts

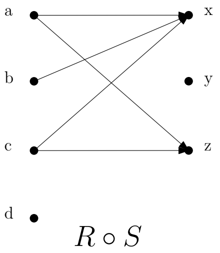


Composing Relations

If $(a, b) \in R$ and $(b, c) \in S$, then $(a, c) \in R \circ S$

- R is a relation from A to B, S from B to C, and $R \circ S$ from A to C
- Draw R and S with the common domain (B)
- Drawing all possible shortcuts





Composing Relations: Exponentiation

A binary relation can be composed with itself

$R^2 = R \circ R$ is defined as

If $(a, b) \in R$ AND $(b, c) \in R$, then $(a, c) \in R^2$

In general,

$$R^n = \underbrace{R \circ R \circ \ldots \circ R}_{n \text{ times}}$$

is recursively defined as

$$R^n = \begin{cases} R & \text{if } n = 1 \\ R^{n-1} \circ R & \text{else} \end{cases}$$

Composing Relations

Let $A = \{1,2,3,4\}$. Consider the following two relations on A

- $R_1 = \{(1,1),(1,2),(2,2),(2,1),(3,3)\}$
- $R_2 = \{(1,3),(3,2),(2,1)\}$

Write the following relations

ICP 6-17 $R_1^2 = R_1 \circ R_1$

ICP 6-18 $R_1^3 = R_1 \circ R_1 \circ R_1$

ICP 6-19 $R_2^2 = R_2 \circ R_2$

ICP 6-20 $R_2^3 = R_2 \circ R_2 \circ R_2$

ICP 6-21 $R_1 \circ R_2^2$

ICP 6-22 $R_2^2 \circ R_1$

Composing Relations: Exponentiation

$$R^n = \underbrace{R \circ R \circ \ldots \circ R}_{n \text{ times}}$$

Let R be a relation on set of people

$$(a, b) \in R \quad \text{if } a \text{ is a parent of } b$$

What is R^2, R^3, ...

Composition of Relations is Associative

Homework: Show that for any three relations R, S, and T

$$(R \circ S) \circ T \;=\; R \circ (S \circ T)$$

$\triangleright$ you just need to use definition of composition

Homework: Show that for any relation R and integer $n > 1$

$$R^n \;=\; R^{n-1} \circ R \;=\; R \circ R^{n-1}$$