Functions

- Ordered tuples and Cartesian Product
- Function and Representations
- Types of Functions
- Composition and Inverse of Function
- Numeric Functions

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Numeric Functions

A numeric function is typically of the form $f : \mathbb{R} \mapsto \mathbb{R}$

In discrete mathematics, generally $f : \mathbb{Z} \mapsto \mathbb{Z}$

The most important property of numeric functions (for us)

Both the domain and codomain are totally ordered sets

$\triangleright$ every pair is comparable, $<, >, \leq, \geq$ are well defined

We can plot $f : \mathbb{R} \mapsto \mathbb{R}$ using the line plot

Monotonic Functions

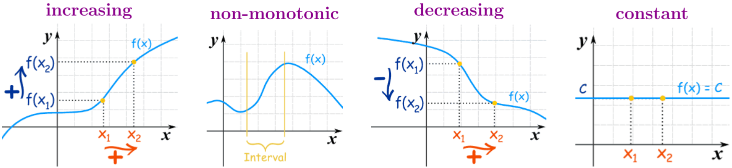
A function $f : \mathbb{R} \mapsto \mathbb{R}$ is increasing if $f(x) \leq f(y)$ whenever $x < y$

$$\forall x \, \forall y \; \big(x < y \; \rightarrow \; f(x) \leq f(y)\big)$$

f is called strictly increasing if $f(x) < f(y)$ when $x < y$

▷ Decreasing and strictly decreasing functions are defined similarly

Collectively, such functions are called *monotonic functions*



source: mathsisfun.com

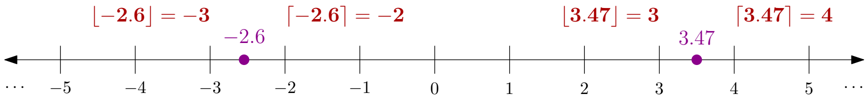
Note: strictly increasing/decreasing functions are necessarily one-to-one

Important Numeric Functions: Ceiling and Floors

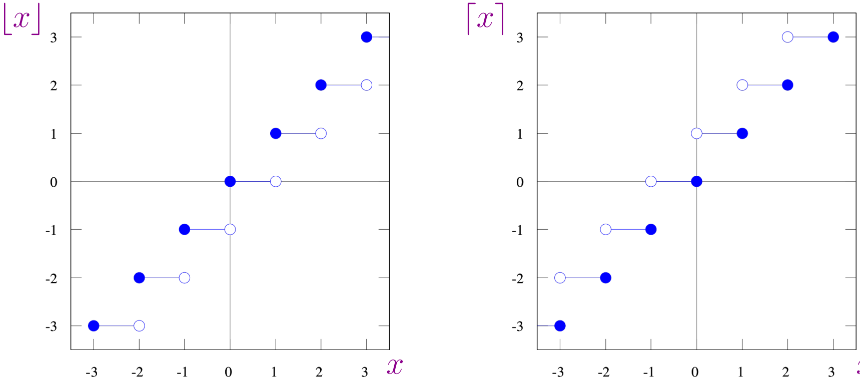
The floor and ceiling functions map real numbers to integers ($\mathbb{R} \mapsto \mathbb{Z}$)

Given $x \in \mathbb{R}$,

- **floor**(x), $\lfloor x \rfloor$ is the largest integer $\leq x$
- **ceiling**(x), $\lceil x \rceil$ is the smallest integer $\geq x$



Important Numeric Functions: Ceiling and Floors



source: Wikipedia

Important Numeric Functions: Ceiling and Floors

ICP 5-30

- $\lfloor 1.7 \rfloor, \quad \lceil 1.7 \rceil$
- $\lfloor 3.2 \rfloor, \quad \lceil 3.2 \rceil$
- $\lfloor 3.5 \rfloor, \quad \lceil 3.5 \rceil$
- $\lfloor 3.9 \rfloor, \quad \lceil 3.9 \rceil$
- $\lfloor 11 \rfloor, \quad \lceil 11 \rceil$

ICP 5-31

- $\lfloor -1.7 \rfloor, \quad \lceil -1.7 \rceil$
- $\lfloor -3.2 \rfloor, \quad \lceil -3.2 \rceil$
- $\lfloor -3.5 \rfloor, \quad \lceil -3.5 \rceil$
- $\lfloor -3.9 \rfloor, \quad \lceil -3.9 \rceil$
- $\lfloor -11 \rfloor, \quad \lceil -11 \rceil$

Important Numeric Functions: Logarithms

- Exponentiation functions and their inverses (logarithmic functions) are very important numeric functions

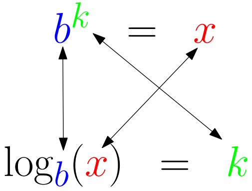
- Appear everywhere in computational finance, algorithms, sciences, probability, information theory and ...

Important Numeric Functions: Logarithms

b^k: What do we get when we multiply b, k times?

$\log_b(x)$: How many of b do we multiply to get x?

$$b^k = x \quad \Longrightarrow \quad \log_b(x) = k$$

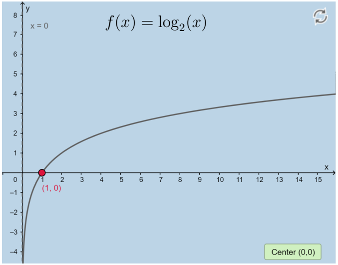


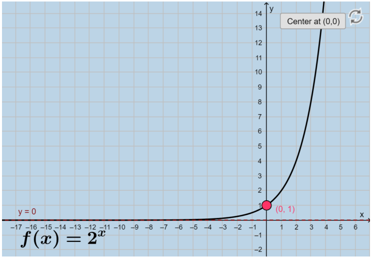
Important Numeric Functions: Logarithms

b^k: What do we get when we multiply b, k times?

$\log_b(x)$: How many of b do we multiply to get x?

- b^k and $\log_b(x)$ are inverse of each other





plots made with https://www.geogebra.org/

Important Numeric Functions: Common Log Bases

$$\log_b(\cdot)$$

- Common Logarithm: Base 10: $b = 10$
  - $\triangleright$ Denoted by $\log(\cdot)$ - generally used in finance, engineering
- Natural Logarithm: Base e: $b = e \sim 2.71828\cdots$ (Euler's number)
  - $\triangleright$ Denoted by $\ln(\cdot)$ - generally used in science, mathematics
- Binary Logarithm: Base 2: $b = 2$
  - $\triangleright$ Denoted by $\lg(\cdot)$ or $\log(\cdot)$ - generally used in computer science

Important Numeric Functions: Properties of Logarithms

These properties of logarithms simplify calculations

1. Log of a product is the sum of logs
   - Convert multiplication into addition
   - $\log(xy) = \log x + \log y$ $\triangleright$ $\because 2^a 2^b = 2^{a+b}$
2. Log of a quotient is the difference of logs
   - Convert division into subtraction
   - $\log \frac{x}{y} = \log x - \log y$ $\triangleright$ $\because \frac{2^a}{2^b} = 2^{a-b}$
3. Log of a power is the power times the log
   - Convert exponentiation into multiplication
   - $\log x^y = y \log x$ $\triangleright$ $\because (2^a)^b = 2^{ab}$

Important Numeric Functions: Change of Bases

Computation of $\log_{b_1}(x)$ can be done using $\log_{b_2}(\cdot)$

$$\log_{b_1}(x) = \frac{\log_{b_2}(x)}{\log_{b_2}(b_1)}$$

Important Numeric Functions: Numerals in numbers

A number n requires $\lfloor \log(n) \rfloor + 1$ digits in writing

A number n requires $\lfloor \lg(n) \rfloor + 1$ bits in writing